MATHEMATICAL NOTES

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Sets vs. Boolean Algebra

Classical propositional logic and set theory are often considered to be two instances of Boolean algebra, with set union corresponding to logical **or**, and set intersection corresponding to logical **and**. However, this does not reflect the logical structure of set theory. Thus, any set may be considered as the union of one-element sets, the sense of this representation depending on interpretation:

- *enumeration* (strong union): a set is treated as element *a* **and** element *b* **and** element *c* **and** …; this makes a simultaneous interpretation of a set as *actual* integrity;
- *sampling* (weak union): a set is treated as element *a* **or** element *b* **or** element *c* **or** …; this interpretation stresses the idea of *potential* integrity, referring to the operation of "probing" the set by random selection of one or another element.

On the other hand, enumeration is algorithmic, in the sense that one is supposed to be able to construct the set being given its elements; on the contrary, the sampling technique refers to the quality of the set, the properties of the elements that make them belong to the set. Compare:

- *categorization by convention*: let us refer to Mr. and Mrs. Jones as the Jones family;
- *explanation by example*: the vegetables are… well… the carrot, the cucumber, the onions, and like.

Similarly, the two types of definition:

- *by construction* (explicit): numeric data types include integer, real, and date-time;
- *by function* (implicit): let all the real numbers $x < 0$ be called negative.

8 Jan 2000

Negative Sets

All fundamental mathematical objects are nothing but an abstract expression of the typical modes of activity. Thus, integer number originated from the procedure of counting and recounting, with the result that happened to never depend on the enumeration order. On the contrary, negative numbers express the idea of the impossibility of counting, of lack, or debt (so that the usual sign multiplication rule becomes related to the custom of debt compensation or paying off). Later on, when the overall idea is clear enough, one can get engaged in constructing formal models; thus number theory come to light, with its peculiar theorems.

The idea of a set is an abstraction of inclusion, of the possible involvement of an object in some human activity. When we set about doing something, we first look at what could be useful for that, and what would impede it. All we see gets evaluated from this viewpoint. This is an entirely qualitative assessment: if it does for our purpose, we'll keep it in the mind; if it won't do, just drop it and don't care any longer. Formally, we speak about an element belonging to a set, meaning our ability of constructive check. A set is exactly the entirety of what in contains, and it has nothing to do with the rest. In other words, it is only the property of belonging that is properly defined, while not belonging is something most uncertain (just because we cannot know everything in the world, including what is yet to come). However, when our activity is a part (a stage) of another activity assuming a wider range of objects to involve, the property of not belonging can be taken in a narrow sense, as belonging to a complement. This is a quite testable hypothesis.

In general, rather complex hierarchical structures can emerge in this way; this reflects the diversity of human activity. Non-belonging is differently defined in different contexts. Still, in any case it refers to belonging to something else. Mathematicians prefer to deal with a constructively defined "universe" and never touch whatever lies beyond. This perfectly matches the natural circumstance that we always work with what we have, which is here at hand and is (at least in principle) available.

However, in real life, besides the things that are fit for the current activity we often encounter things that are incompatible with it, or practically inaccessible. With that which is somehow present but cannot be used (is "forbidden"). Such objects are essentially related to the activity, they belong to it as well, but in a "negative" manner. These are not elements, but rather *holes*, the indicators of the necessity to exclude some things from consideration.

The term has been borrowed from physics, where electrons and holes productively co-exist in atomic models, in semiconductors, and in many other practically important areas of research. In a way, a positron is nothing but a hole in the vacuum produced by the ejection of an electron; that is why electrons and positrons are always born in pairs. We find that the idea of a hole leads to a practically attractive mathematics in set theory.

As we already know that the complement may play the role of subtraction for sets, we can formally define a negative set as the complement of a regular set to the empty set, as its subtraction from "zero". The existence of such a complement can be simply postulated. We express it as $(-)A = \varnothing \setminus A$. The minus sign has been put in parentheses to stress its operator nature. By definition, every element of the negative set is a hole in the position of a presumable element of the original set. Obviously, $A \cup (-)A = \varnothing$. That is, considered together in the same activity, the element and the hole will "annihilate" each other, so that we have neither inclusion, nor exclusion. Symmetrically, $A = \varnothing \setminus (-)A = (-)(-)A$. In this way, we fix the logic of the theory; in some activities, this condition may not necessarily hold.

Observing that sets are not linearly ordered like numbers, we can extend the study of regular and negative sets to the general case of arbitrary collections of elements and holes; for simplicity, we keep calling them sets (or classes, if you wish). Clearly, a general set can be decomposed into the "positive" and "negative" parts:

$$A = A_{\mathrm{p}} \cup (-)A_{\mathrm{n}} ,$$

where A_{p} and A_{n} are regular sets (just elements, without holes). In particular, they may be empty. In a union of two sets, the corresponding elements and holes annihilate:

$$A \cup B = (A \cup B)_{\mathrm{p}} \cup (-)(A \cup B)_{\mathrm{n}} ,$$

with

$$(A \cup B)_{\mathrm{p}} = A_{\mathrm{p}} \setminus (A_{\mathrm{p}} \cap B_{\mathrm{n}}) \cup B_{\mathrm{p}} \setminus (B_{\mathrm{p}} \cap A_{\mathrm{n}})$$

$$(A \cup B)_{\mathrm{n}} = A_{\mathrm{n}} \setminus (A_{\mathrm{n}} \cap B_{\mathrm{p}}) \cup B_{\mathrm{n}} \setminus (B_{\mathrm{n}} \cap A_{\mathrm{p}})$$

A less trivial mathematics comes in as we pass from addition to multiplication, from the union to the intersection of sets. Here, formal evaluation requires the introduction of the common sign multiplication law: $(+)(+) \to (+)$, $(-)(-) \to (+)$, $(+)(-) \to (-)$, $(-)(+) \to (-)$. The double negation has already been mentioned; basically, it can be translated as "the absence of absence is presence". In the same manner, we understand both the presence of absence and the absence of presence as absence. Once again, this is not always so in real life, but it will do in many importance cases. Then, obviously,

$$A \cap B = (A_p \cup (-)A_n) \cap (B_p \cup (-)B_n) = ((A_p \cap B_p) \cup (A_n \cap B_n)) \cup (-)((A_n \cap B_p) \cup (A_p \cap B_n))$$

Thus, one could construct a special case of an “antiset” of an arbitrary set:

$$(-)A = \varnothing \setminus A = (-)(A_\mathrm{p} \cup (-)A_\mathrm{n}) = A_\mathrm{n} \cup (-)A_\mathrm{p}$$

From the quantitative viewpoint, regular sets are characterized by the number of elements, assuming that each element contributes +1 to the total. Negative sets obviously imply the number of holes, each contributing –1 to the number of elements, so that the potency of a negative set is negative. For a general set, one needs to sum the positive and negative contributions, and any combination is possible.

In applications, holes may become a promising formalization of the notion of need, which is practically important in sciences like psychology or economy. On the other hand, noting that particle annihilation in physics leads to the production of new particles, some quite nontrivial theories of activity could be developed. By the way, the notion of element is one of the most elusive in the present set theories; one can hardly tell it from a set. Traditionally, mathematicians tend to identify elements with sets thus limiting themselves to narrow range of the possible theories. One could observe that any object as an element is nothing but the class of all the sets containing it; conversely, as a hole, any object is associated with a class of sets it does not belong to.

Any mathematician will readily indicate the algebraic structures covering the above, and proceed to the far-fetched conclusions. Here, we are not interested in the formal details; we rather seek for an intuitively pleasing notion of qualitatively distinct mathematical object. Such fundamental objects cannot be reduced to each other, even resembling each other so much. Thus, one could derive mathematical logic from set theory, or vice versa; the both will keep their special ways. Real numbers modelled with convergent sequences of rational numbers will remain real; even if we identify them as the elements of a field, this will result in yet another (general algebraic) model, abstracted from any other (possibly as important) aspects. Real numbers do not come from math; they come from praxis. Just like integers, or complex numbers, or geometric shapes, or sets.

Now, we get negative sets. Proceeding in this direction, one can construct complex sets, or spaces of an arbitrary dimension… Let it be postponed to a better time.

Dec 1984

Fuzzy Sets and Relativistic Velocity Addition

The axiomatic skeleton for fuzzy set intersections $i(a, b)$ and unions $u(a, b)$ is given by:

1. boundary conditions:

$$i(1, 1) = 1$$
$$i(0, 1) = i(1, 0) = i(0, 0) = 0$$
$$u(1, 1) = u(0, 1) = u(1, 0) = 1$$
$$u(0, 0) = 0$$

2. commutativity:

$$i(a, b) = i(b, a)$$
$$u(a, b) = u(b, a)$$

3. monotonicity:

if $a \le a'$ and $b \le b'$ then $i(a, b) \le i(a', b')$ and $u(a, b) \le u(a', b')$

4. associativity:

$$i(i(a, b), c) = i(a, i(b, c))$$
$$u(u(a, b), c) = u(a, u(b, c))$$

In general, fuzzy set unions and intersections are not idempotent. Every possible choice of the form of fuzzy union, intersection and complement would violate at least some properties of the Boolean lattice. In particular, the law of excluded middle and the law of contradiction may not hold.

As it is well known, the conditions 1–4 lead to inequalities

$$i(a, b) \le \min(a, b)$$
$$\max(a, b) \le u(a, b) ,$$

so that $i(a, b) < u(a, b)$ for $a \ne b$. The most common form of fuzzy set complement is $c(a) = 1 - a$.

One could easily construct an example illustrating that the axioms 1–4 may be incompatible with DeMorgan's laws:

$$u(a, b) = c(i(c(a), c(b)))$$
$$i(a, b) = c(u(c(a), c(b)))$$

Indeed, let us assign to every element x_i of a fuzzy set $A = \sum_i \mu_i x_i$ a number $\chi_i \in (-\infty,+\infty)$ such that

$$\mu_i = \tfrac{1}{2}\left(1 + \text{th}(\chi_i)\right).$$

With χ varying from $-\infty$ to $+\infty$, μ monotonically increases from 0 to 1. Let the complement of A be constructed according to the usual rule:

$$c(\mu(\chi)) = 1 - \mu(\chi) = \tfrac{1}{2}\left(1 - \text{th}(\chi)\right) = \tfrac{1}{2}\left(1 + \text{th}(-\chi)\right) = \mu(-\chi) .$$

Finally, let us define the intersection of two fuzzy sets as

$$i(\mu_1(\chi_1), \mu_2(\chi_2)) = \mu(\chi_1 + \chi_2) = \tfrac{1}{2}\left(1 + \text{th}(\chi_1 + \chi_2)\right) = \mu_1\mu_2 / (1 - \mu_1 - \mu_2 + 2\mu_1\mu_2) .$$

This expression satisfies the boundary conditions for fuzzy set intersection; it is evidently commutative and monotonic. Direct computation shows that it is also associative:

$$i(i(\mu_1(\chi_1), \mu_2(\chi_2)), \mu_3(\chi_3)) = i(\mu_1(\chi_1), i(\mu_2(\chi_2), \mu_3(\chi_3)) = \frac{\mu_1\mu_2\mu_3}{(1 - \mu_1 - \mu_2 - \mu_3 + \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)}$$

Thus defined $i(\mu_1, \mu_2)$ is however not idempotent, though it is *asymptotically* idempotent at $\chi \to \pm\infty$, which ensures a correct transition to ordinary (crisp) sets.

Now, let us use DeMorgan's law to introduce set union through intersection and the complement:

$$u(\mu_1(\chi_1), \mu_2(\chi_2)) = c(i(c(\mu_1(\chi_1)), c(\mu_2(\chi_2)))) = c(i(\mu_1(-\chi_1), \mu_2(-\chi_2))) =$$
$$= c(\mu(-\chi_1 - \chi_2)) = \mu(\chi_1 + \chi_2) = i(\mu_1(\chi_1), \mu_2(\chi_2))$$

Since thus defined union and intersection coincide, set union will not satisfy the axiomatic skeleton, violating the boundary conditions at $\mu_1 = 0$ and $\mu_2 = 1$. And, of course, the value of $i(\mu_1, \mu_2)$ can never be less than $u(\mu_1, \mu_2)$ as it should be from the skeleton axioms. It should be noted, that the law of excluded middle and the law of contradiction are not satisfied for this choice of intersection/union:

$$i(\mu(\chi), \mu(-\chi)) = \tfrac{1}{2}$$

The physical sense of the above definitions is relativistic addition of velocities. The particles moving forward with the speed of light are associated with $\mu = 1$, while those moving with the speed of light in the opposite direction are associated with $\mu = 0$. Note that the definition of union based on the addition of only positive velocities

$$\mu(\chi) = \text{th}(\chi_1 + \chi_2)$$

$$u(\mu_1(\chi_1), \mu_2(\chi_2)) = \mu(\chi_1 + \chi_2) = \text{th}(\chi_1 + \chi_2) = (\mu_1 + \mu_2) / (1 + \mu_1\mu_2)$$

is known as a function of the Hamacher class with $\gamma = 2$, and, together with the standard complement, it produces the intersection

$$i(\mu_1, \mu_2) = \mu_1\mu_2 / (1 + (1 - \mu_1)(1 - \mu_2)) ,$$

so that both the axiomatic skeleton and DeMorgan's laws are satisfied. However, in this case, the definition of the complement is "physically incompatible" with the definition of the union, since Galilean velocity addition principle is mixed with the relativistic rule.

14 May 1997

Logical Symmetries and Complex-Valued Logic

Traditionally, the algebraic formulations of logic would represent the logical value **true** with the number 1, and the logical value **false** with the number 0. Then the operations of conjunction ($\wedge$) and disjunction ($\vee$) get naturally represented by algebraic multiplication and addition respectively. The rules of thus defined algebra significantly differ from common arithmetic; still, operating with numbers rather than special logical values makes certain aspects of logical theory more transparent, allowing for many productive generalizations.

There is a different approach that might sometimes come handy in logical development. Instead of representing the logical value **false** with 0, one could rather represent it with the number –1. Denoting the logical negation with the minus sign, we can naturally obtain –1 (**false**) as **not true**, while –(–1) equals 1 by definition:

$$-(-a) = a\,,$$

with the letters like a, b, c, … denoting, by convention, anything evaluating to a logical value: a constant, a variable, or a logical expression (formula) containing constants, variables and other expressions in any combination; further, one can consider logical-valued functions as abbreviations for some kinds of expressions and move on to variable functions and functional formulas; for our algebraic consideration this hierarchy is not really important.

Now, let us introduce algebraic multiplication * so that 1 * 1 = 1 and (–1) * (–1) = 1, with the natural choice of (–1) * 1 = 1 * (–1) = –1. This operation obviously corresponds to logical equivalence, and it is idempotent, commutative and associative:

$$a * b = b * a$$

$$(a * b) * c = a * (b * c)$$

but, unlike conjunction and disjunction, it is not idempotent:

$$a * a \neq a$$

which does not seem to make much trouble, since the common numeric multiplication is not idempotent either. One can also observe that

$$-(a * b) = (-a) * b = a * (-b)\,,$$

which pleasingly complies with our the arithmetic habits.

Thus introduced multiplication is symmetrical in respect to truth valuation: swapping the truth values $1 \leftrightarrow (-1)$ will produce the same valuation table for logical equivalence.

It should be stressed that the equality sign in the above expressions, albeit a kind of equivalence too, belongs to the logic of our description (methodological level) rather than to the logical structures to describe. The two levels may differ in their algebraic structure, including the very specification of the logical values. We may use equality and inequality to compare the formulas of the logic in question, but not within any logical formula.

Using the above equations, we can formally eliminate negation from our theory replacing it with multiplication:

$$-a = (-1) * a\,.$$

This opens a promising direction of thought, since logical negation is a very specific operation that has historically raised much controversy. It is not quite obvious why we should stick to it in multi-valued or fuzzy logics; all we can refer to is a long-lived tradition. However, as one can see, negation is not entirely eliminated in this approach, since we have to prefer one logical value to another; this is just another way to say the same. Alternatively, one could treat non-equivalence as an independent binary operation:

$$a \times b = -(a * b),$$

which implies the introduction of the "complementary negation":

$$\sim a = (1) * a$$

and thus a completely balanced theory with no unary operations. The applications of such a logic are to stress the practical polarity of the activities establishing equivalence and those that seek for distinction, with a wide range of intermediate modes.

Obviously, the truth-values of this symmetrized logic are expressed through the two basic operations:

$$1 = a * a$$

$$-1 = a \times a$$

for any *a*. However, since this definition implicitly introduces a quantor, it belongs to a higher level of hierarchy, and we cannot employ the two logical values in our algebraic logic unless we do it in a symmetrical manner, like in the example of complementary negations. In other words, the space of truth-values is thus understood as global in respect to the object area: there may be many logical values, but each of them would represent a statement about the whole of it. Still, when we mean some particular statements about the object area, we can expand the two constants as above, thus restricting ourselves to the common two-valued logic.

A fully symmetrical logic could be enriched by the introduction of asymmetric operations that make difference between falsity and truth, so that their valuation tables would no longer be the same after replacing falsity with truth, and *vice versa*. The two common examples are provided by conjunction and disjunction, which are interrelated through either equivalence or negation:

$$a \wedge b = -((-a) \vee (-b))$$

$$a \vee b = -((-a) \wedge (-b))$$

One could treat these dual operations as "additive", compared to the multiplicative nature of equivalence and non-equivalence. In a logic without negation, they become independent, just like equivalence and non-equivalence. Still, in the binary case (that is, with implicit negation), we can symmetrically rewrite them as

$$a \wedge b = ((a * a) \times (b * b)) * (((a \times a) * a) \vee ((b \times b) * b))$$

$$a \vee b = ((a * a) \times (b * b)) * (((a \times a) * a) \wedge ((b \times b) * b))$$

or in any other similar form. Additive (asymmetric) operations might be considered as more fundamental, compared to equivalence and non-equivalence, since one could construct symmetrical combinations of asymmetric operations, but not the other way round:

$$a * b = (a \wedge b) \vee ((\neg a) \wedge (\neg b))$$

$$a \times b = (a \vee b) \wedge ((\neg a) \vee (\neg b))$$

Hence, the formalism of binary logic can be constructed from just two operations: logical negation and some asymmetric ("additive") operation. However, this simplicity is delusive, as it is entirely based on the idea of logical negation, to compensate one asymmetry with another. In some cases, this original constraint may get too restrictive.

Seeking for generalizations, let us recollect Peano's *Arithmetices Principia*. Leaving aside the artificiality of his set-theoretic formulations, we stress, that natural numbers start with postulating the existence of a *unit*, so that all the other natural numbers could be expressed in terms of this primary quantity. Modern mathematicians are inclined to treat the pseudo-number 0 as a natural number, which breaks the theory's transparence and consistency. Yet another important issue concerns Peano's introduction of numerical equality; later authors have identified it with logical equivalence, which is certainly inconsistent, confusing quite different levels of hierarchy just because they apparently "isomorphic". Natural arithmetic could then be derived from the unity, equality, and a single unary operation, increment (which, in general, is not the same as binary addition of a natural number and unity). However, this inductive construct did not immediately reveal the inner symmetries of the theory, and the definitions of addition and multiplication were introduced, which then lead to the notions of subtraction and division alien to the natural domain. Subtraction lead to the idea of negative numbers and positioned zero as an integer. Division produced rational arithmetic as a basis of real numbers. The same development is also possible in logic. However, here, we are interested in yet another extension, complex-valued logic.

We recall that the space of logical values in the binary limit is structured so that $(-1) * (-1) = 1$. Now, let us similarly introduce a logical value that would represent the square root of logical falsity: $i * i = -1$. Well, we do not yet know what it really is and how it can be produced, but we can simply postulate its existence and denote it with the character i. The possibility of such abstract conceptualizations has once become the cornerstone of all the modern mathematics. However, in our algebraic logic this *imaginary* unit value has a quite straightforward interpretation. Indeed, the equality $a * a = 1$ means that every real logical entity is equivalent to itself, that is, $a \equiv a$ is **true** (where we mean the object area equivalence rather than the logic of its description). The equality $i * i = -1$ will hence introduce an entity that *is not identical to itself*. However strange this may sound for a scientifically minded person, the idea is not entirely new; it has long since been speculated upon in philosophy (especially in Hegel's system and in Marxism). That is, our algebraic formulation of the classical logic might become a good starting point for the reconciliation of the scientific and philosophical ways of thought provided complex truth-values are taken into account.

In this extended logic, every truth-value may contain both a real and imaginary component. Depending on the choice of the "additive" operations, one will obtain different rules for arithmetic manipulations with complex truth-values. In general, these rules will be much more complicated than in conventional complex arithmetic. This does not deny the correspondence in principle. For instance, the traditional interpretation of phase as vector rotation angle immediately applies to logical negation interpreted as vector inversion. This also leads to the distinction of inversion and mirror reflection, revealing some fine details of logicality.

The presence of the "phase" component in any logical value opens new perspectives for quantum logic, since the "observable" reasoning is restricted to real numbers, hiding the inner phase-dependent derivations. The introduction of such hidden logical states is similar to the transition from the observable position and momentum of a classical particle to the virtual motion in the inner configuration space of a quantum system that do not directly correspond to any observable quantities.

To conclude, the symmetric form of algebraic logic is promising enough to suggest a closer look to its generalized versions, including many-valued, rational, fuzzy and complex logic. It becomes especially appealing as, on the one hand, the kind of mathematics thus obtained does not demand a revolutionary shift of paradigm, and on the other hand, the new notions remain quite intuitive and tractable even for those who do not praise much the excess of formal sophistry. The hierarchy of logic has yet much room for nontrivial development.

Nov 2003

Points and Limits

Traditionally, the typical procedure of constructing a metric space S looks like that: let us take a set B (which will be called *a base* in the following; her, we don't discuss topology, and there should hardly ever be any confusion), and let us know how, *for any* two points x and y from B, to find a (real) number ρ, to be called the distance between x and y, provided the following conditions are satisfied:

(1) $\rho(x, y) = 0$ if, and only if $x = y$
(2) $\rho(x, y) = \rho(y, x)$
(3) $\rho(x, y) \leq \rho(x, z) + \rho(x, z)$

In the above, the same quantifier “for any” is implicitly meant (and we do not yet ponder much over its sense and feasibility). The second axiom, to be true, just says that metrics is a function of a *subset* rather than an ordered pair, as the order of elements does not matter. This makes one think that thus defined distance is basically a kind of measure: it hides the size of what lies between the end points (the length or duration to walk). This too requires a separate treatment. The last property is commonly known as the triangle inequality; it is here, where most alternative theories of distance introduce their specific deviations (for instance, like in ultrametric spaces).

The formal mathematics holds that one is free to define anything in any way. Nobody cares for the reason. In reality, of course, definitions are never spun out of thin air: they are intentionally designed to produce what we need in the end. Since this ultimate goal grows from the common practice, arbitrary solutions are very unlikely to encounter. A layperson, however, may miss the inner sense of what is going on, and this makes it hard to get through the rest of the science, however logically derived from the primary notions. Typically, mathematicians are not inclined to put things plain; even worse, they try hard to disguise the practical motivation. Here, we’ll look closer to what is usually left in the shade.

On the next stage of metric theorization, one comes to studying sequences of points from *B*, which are formally defines as some mappings of the natural numbers into the base. However, in fact, we need an algorithm that would allow us to choose one point of the base after another; it is this practical principle (along with a starting point) that defines the resulting sequence. Devoid of this systemic directedness, the collection of points is not really a sequence, but rather a mere set. One could easily observe that the very construction of the natural numbers is exactly like that: as soon as you reach a certain mark, go on to the next. The common arithmetic operations are introduced later on, formally imposed as an optional structure due to one of the possible ways of identifying different sequences. Well, let the topic wait for a better season.

So far, we are left with the succession of base points, which is traditionally denoted using the subscript notation: x_n , with $n = 1, 2, \ldots$ (sometimes, starting from zero, or from an arbitrary positive number). Sequences may be of any sort, and the same points may be counted many times. In the trivial case, the whole sequence will contain a single point of the base, stubbornly reproduced at each sampling. In general, a sequence may exhibit multiple self-intersections, numerous loops. The important special case of this generalization is provided by periodic sequences (closed orbits). Eventually, sequences may (appear to) be random; this is yet another reason to avoid considering sequences as instantly given entities, to honestly compute them once needed, without any pretense to get the same result the next time.

Since we are interested in distances, we immediately discover two complementary modes of transforming sequences of base points (which may be complex and poorly tractable, or even not mathematical at all) into sequences of numbers (which seem to be much more familiar). First, we can compute the distances between the elements of the sequence: $\rho_{\text{in}}(n; m) = \rho(x_{n+m}, x_n)$. Alternatively, one could fix a point in the base and determine the distances of all the elements of the sequence to this reference point: $\rho_{\text{out}}(n) = \rho(x_n, x_0)$. The notation is to stress the different character of these quantities: either inner or outer structure. The former is well known from mathematical statistics, as a variety of autocorrelation. The complete collection of such functions (with all possible *m*) may be considered as a fait account of the inner organization of the sequence, regardless of its embedding in the incident space. The latter construct puts us in the framework of vector analysis, so that any point of the base could be represented by its radius vector; with a few reference points to start with (the required number depending on the nature of the base and the way of its arithmetization), we can specify the direction as well. When such multiple reference points form an independent sequence, we come to a “synthesis” of the inner and outer structures, sequence comparison: $\rho_{\text{y,x}}(n; m) = \rho(y_{n+m}, x_n)$, or, conversely: $\rho_{\text{x,y}}(n; m) = \rho(x_{n+m}, y_n)$.

The outer structure, in general, does not follow from the inner structure, and the other way round. This depends on both the organization of the base and the definition of the distance. Still, in many practical cases, the two structures seem to lead to basically the same view.

One of the most important ideas related to such correspondence is provided by the notion of convergence. Thus, if, for any positive real number ε, there is a natural number N, such that $\rho_{\text{in}}(N; m) \le \varepsilon$ for all $n \ge N$ at some fixed m (usually set to unity), the sequence x_n is called a Cauchy sequence. Alternatively, if, for any ε, there is some N, such that $\rho_{\text{out}}(n) \le \varepsilon$ for all $n \ge N$, we say that the sequence of points x_n *converges* to the point x_0 , or, equivalently, that the point x_0 is the *limit* of the sequence, which is commonly written as $x_n \to x_0$. Convergence of the points of the base thus gets reduced to convergence of real numbers abstracted from their object area. This may lead to spurious effects, as the properties of numeric sequences do not exactly correspond to the properties of the objects of interest, and an imprudent judgement neglecting the essential features of the object area may lead to logical fallacies.

In metric spaces, the triangle rule makes every converging sequence a Cauchy sequence as well. The converse is not true, as there may be no point of the base infinitely close to the sequence points. On the other hand, the same triangle rule implies that, with $x_n \to x_0$ and $y_n \to x_0$, also $\rho_{\text{y,x}} \to 0$. One is tempted to believe the converse to be true: if $\rho_{\text{y,x}} \to 0$ then x_n and y_n either simultaneously fail to converge or converge to the same base point. If this were so, one could boldly consider all the sequences with $\rho_{\text{y,x}} \to 0$ as equivalent, so that the base could be completed by such equivalence classes taken for the lacking limit points. In elementary mathematics, we simply observe that, if the distance between the limit points of equivalent sequences is non-zero, it is enough to choose ε equal to the half of that distance to make the three convergence conditions (for x_n , y_n, and $\rho_{\text{y,x}}$) violate the triangle rule; consequently, the distance between the limit points must be zero, and then by the first rule of metrics, the limit point will coincide, which seems to be the desired result.

I dare to make a boring suggestion: let us pierce the inviolable wall of mathematical rigor and look out through the tiny hole into the open of not so elementary world. Logically, the definition of the limit only says that the distance between the limit point and the elements of the sequence *can be made* smaller than any fixed real number. But this is not an identity, unless we deal with the same point infinitely repeated. Similarly, the parallel convergence to two different point means that the distance between these limits *can be made* smaller than any number, and not that the distance *is* zero. In other words, the distance between the limit points of two equivalent sequences is the limit of a numeric sequence rather than a ready-made real number. It *tends* to zero, but does not *equal* zero. In the early days of mathematical analysis, its founding fathers spoke of *infinitely small* quantities, never identifying them with real numbers. Later, the static paradigm has expelled the notion of an infinitely small value from any school courses; in the XX century, the term has been revived in the context of nonstandard analysis (which, however, merely tried to tame the essentially dynamic idea reformulating it in the same static language). Now, the distance between the elements of converging sequences is infinitely small, but it is not zero. Logically, we cannot apply the first rule of metrics to such quantities, except for a few special cases (say, the isolated points of the base).

For the same reasons, the triangle rule does not apply to the convergence process, being initially coined for finite (static) quantities. A similar rule concerning infinitesimal values would look differently: $\rho(x, y)$ is an infinitesimal of the same or higher order compared to $\rho(x, z) + \rho(x, z)$.

Let's dig a little bit deeper. When we compare two sequences, we leave the initial metric space S to arrive at a *different* space, with the base composed of Cauchy sequences on S. The equivalence of sequences is defined in respect to that new space. So, the zero distance as the measure of the difference of sequences is not the same as the zero distance between the elements of the initial base, and we have no right to compare x_n , y_n, and $\rho_{\text{y,x}}$ within the same triangle rule! This is a logical fallacy, term substitution. We have not accounted for the fact that the same number (name) may label qualitatively different entities. Distances in the space of sequences may be computable on the basis of distances between their elements; still, these are different notions of distance, whether their numerical values coincide or not.

To identify a class of equivalent Cauchy sequences in the space *S* with a point of its base *B*, one needs a special operation, which does not need to be always feasible, and which may be equivocal. For instance, consider a random identification with the points of some area in *B*, as described by a probability distribution. It is only in a very special case, when the distribution is represented with the δ-function (which, as we know, is not entirely a function but rather a functional), that the projection would give a kind of a point. In the same lines, if a Cauchy sequence does not converge to a base point, we cannot formally complete the base adding a new point, since such additional points may be logically incompatible with the object area of the theory and need a different theory, with a different base.

And finally, for advanced dummies. The collection of sequences converging to some base point *x* could be treated as its infinitesimal neighborhood. Each point therefore becomes a center of the cloud of infinitely small deviation from that point, its virtual variations. For every positive real number *r*, the number of elements in any sequences converging to *x* that remain outside the ball of the radius *r* is finite. Provided, in a meaningful theory, the sequences are constructed according to the same generic principle, one could estimate the average number of points $-\varepsilon(r)$ outside the *r*-sphere; the sign has been chosen to reflect the fact that any sequence is only “shortened” at any level *r*. Those acquainted with nonstandard analysis may invoke a kind of ultrafilter. The quantity $\varepsilon(r)$ serves as a measure of connectedness of the point *x* to the base *B*, resembling the binding energies of electrons in an atom or ion. This “energy” is negative due to a special choice of the reference level: we count from the threshold of detaching the element from the set (the analog of the ionization potential in atomic physics). Depending on the structure of the base (the object area of the theory), one will obtain different distributions of such binding energies. Thus, in atoms, we often observe series of discrete levels converging to the ionization threshold.

This can readily serve as a basis of the possible generalizations of the notion of an element’s belonging to a set. Normally, an element either belongs to a set or not. Fuzzy set theories admit incomplete (partial) belonging; multiple belonging is an obvious extension. However, there are no indications of the origin and possible forms of the membership functions. Here, we relate membership (the way the element is linked to the set) to the structure of the object area selecting the possible trajectories. Every membership function then becomes a property of the infinitesimal neighborhood of a point and it can be numerically evaluated as an average of some operator acting in this inner space.

That’s the point. Any superstructures of the base set produce a higher-level entity. In the same time, their presence means the development of an inner structure within a base point, its inner space. Thus mathematical object become hierarchies.

It is understood that converging sequences are not the only option available. One can consider any trajectories approaching the limit point, including continuous curves (like all kinds of spirals). Alternatively, one could speak of randomly selected approximations. Additionally, there are various non-explicit definitions (like set intersections). One could even associate points with algorithms or physical processes, with their specific symmetries (“spinor” components). The inner space of a point can be extremely complex, while the base retains a simple metric structure, and its points still coincide for zero distances. With all that, infinitely small distances do not imply anything until we indicate the level of discrimination, unfolding the hierarchy in a specific manner. The transition from the inner dynamics of each point to the bas-level properties requires a definite projection procedure (just like quantum mechanics represents the observables with operators).

We encounter inner spaces every time we are to decide on equality (equivalence) of one thing to another. Quantitatively, this means that some measure of difference tends to zero. In a rough overall comparison, the complexity of the objects to compare is not apparent. Still, as soon as we eliminate distinctions at some level, we need to deal with finer variations, penetrate the “inside” of zero. One does not even need quantum mechanics: it is enough to recall that the Solar system, with all its planetary richness, looks like a single point for the inhabitants of the nearest stars, nothing to say about distant galaxies.

For a purely mathematical illustration, take the equality of complex numbers. Formally, two complex numbers are equal when the distance between them (the absolute value of the difference) is zero. However, if a complex number is specified by the absolute value and phase, the points $\{0, \varphi_1\}$ and $\{0, \varphi_2\}$ are far from being the same. That is, zero distance does not mean perfect coincidence, as we need to approach zero by similar trajectories to ensure the equality of phases. Traditionally, nobody cares for such nuances, and the phase of complex zero is said to be undefined. In other words, the distance in complex plane is defined up to a phase factor, or, alternatively, as a phase average, so that, to be explicit, we have to explain how we compute that average and why.

School mathematics takes the linear algebraic form of the complex number with component-wise equality for primary. In this picture, zero is a single point, like any other. Similarly, a single infinite point is introduced (in projective geometry); however, in real calculations, we have to specify the way we pass these singularities. It is never possible to think of zero and infinity as ordinary numbers; this inclusion is purely conventional. They are not entirely numbers, as they do not exactly behave like numbers.

There is an obvious parallel with vector spaces: a vector as a directed quantity is not entirely the same as its coordinate representation. Where the zero vector is directed?

Can we rationally explain why the linear scheme should be the origin of all? Just admit that rotation might be much more important in the real world than mere translation. Why not? The component-wise representation may be considered as a special case, just how a straight line is a special case of general curve, along which the inner space of each point gets mapped onto the inner space of the next. The standard theory of metric spaces remains true, but only in the zero approximation.

Aug 1988

Hierarchical Dimension

Since mathematics has forcefully abstracted itself from our everyday experience and restricted itself to entirely formal issues, we can no longer comprehend what it really is, space. Almost anything can be referred to as space nowadays. Finally, mathematicians just abandoned this notion and stack to highly formal constructions with the names containing the word “space” by mere tradition, mainly in the meaning of "manifold". For such abstract objects, a few as abstract definitions of dimension have been introduced, which tell nothing to the heart of a regular person. We are to blindly believe in the stories suggested by the big science, and to be content with following their recipes, pushing the keys in a prescribed order nobody knows why and for which reasons. Given that modern physicists tend to meditate over their formulas rather than take notice of nature, one does not expect any clarifications from that party neither. And, of course, there is no use appealing to science-blinded philosophy for elementary coherence.

Still, there are those who have retained a bit of human curiosity and who sometimes want something palpable, tractable with a kind of intuition about the real thing around us (and the real people dealing with such things) rather than mere combinations of ideograms. Our practical notion of space refers to real world. Yes, life is complex and diverse, and one needs to differently arrange for certain effects; this leads to the thought of numerous spaces, each organized to support some specific activity. In certain cases, such spaces will basically differ by a (quantitative) parameter that we associate with the notion of dimension. However, it is a true notion that we need, that is, a variety of common techniques of constructing spaces of any dimension; don’t feed us mere symbolic manipulation.

Let us try to (at least schematically) outline one of the possible solutions. Admitting that any choice requires a lengthy justification, let us, however, start with preliminary hints to the very things to be justified.

In respect to human activity, the idea of space characterizes the available options, the ability of choice. This is how any "spatial" language is used in the everyday life, and life in science does not add any principal difference. So far, the examples from science are more common, as modern philosophy does not accept other authorities. Well, let it be science, with a distant aim of eventually adjusting thus acquired practical experience of space construction to all spheres and levels of activity. Now, let us gradually accumulate the necessary instrumentation.

To make the discussion meaningful, we declare that space does really exist, that it is not a personal freak or sheer fantasy. The world is made that way. And this is how we act in this world. With all that, the existence of space is not of the same kind as the existence of any material things. Space does not exist on itself, without anything at all; space is primarily a relation between things. In philosophy, such matter-dependent existence is called ideal. Conversely, things do not exist regardless of their interrelations, so that any matter assumes some ideality (and this not necessarily space). Under certain conditions, the ideal entities get *represented* by material things. The word "space" is a mere sound, or pigment on paper, or a bright dot on the screen; as soon as we start practically dealing with space (including theoretical discussions), this things (the word) becomes a conventional designation for space within the current activity or the current topic. Space is objective; still, in every particular case, we approach the idea from one of the possible directions. Any features we discover may refer either to space as it is (its "inner organization" that does not depend on our subjective moods), or to some specific implementation of space in our activity ("realization"). It should be stressed that, in addition to the distinction of the notions of different types (levels), each individual notion develops a layered structure of its own. In this context, we distinguish the natural ("geometrical") dimension of space and its outer ("topological") dimension.

0. Point

To be honest, this is not an appropriate idea to start with. Rather, that is what we reach in the conclusion: the summit, the highest degree of abstraction. Still, since this text merely presents something earlier thought up and over, one can afford beginning with the end.

For a constructive theory of dimensionality, a point is the nothing we use to produce anything at all: the vacuum, "zero-dimensional" space, that is, the absence of spatiality as such. The utter impossibility of motion nor action.

As we accept the objectivity of space, a point is the expression of this objectivity. Space contains (or is built of) some points; this is nothing but the affirmation of existence and a specific quality. It is only in respect to its "embracing" space that a point can acquire any definiteness; the point just borrows (inherits) it from its space. That is, there are no points as such: all we have is different spaces that can, in certain contexts, be folded into a point, preserving the same spatial quality.

1. Dimension

In philosophy, there is a category which usually goes under the name of "measure" (not to confuse with the narrow mathematical notion of the same name). The category refers to the very possibility of comparing that to that, when one thing becomes a gauge for another, the unit of measurement. Obviously, what we measure must, in some respect, share the same quality with the chosen unit (that is, be commeasurable with the reference thing). On the other hand, it must differ from the unit, to allow any comparison at all; such distinctions are called quantitative.

Unlike a point, any dimension implies the possibility of motion within certain limits (a "degree of freedom"). So, that is what we call a (one-dimensional) space. For a different choice of the unit, the spatial relations will be expressed by some other numbers (and maybe not numbers at all), which does not influence the objective nature of these relations; the (inner) directedness of the space is as objective, providing a sound basis for the very definition of a single dimension.

Admit that there are several different measures (with the corresponding units of measurement). In this case we speak about a many-dimensional space. In the following we are to discuss the possible interdependencies of the space's dimensions. Here, we observe that, in general, the different dimensions of a space are qualitatively different, and one cannot just add one value to another. For instance (anticipating further discussion), to construct a many-dimensional metric space, one needs to somehow bring the different units to a common measure; in the form for the interval,

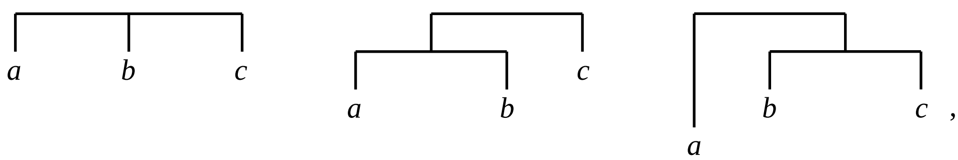
$$ds^2 = g_{ik} dx^i dx^k ,$$

the coefficients g have (physical) dimension

[unit of interval] / [unit i] / [unit k]

Expressing all lengths, say, in meters, we keep in mind a *practically available* procedure of converting the primary units to the desirable result; in the language, the original units often have different names, such as "a running meter", "width" (or "breadth"), "height", and the lots of other names, depending on the kind of what we measure. Consequently, the integrative unit can only be meaningful in the context of an activity requiring that very dimensionality; thus, there is no use to convert all currencies to US dollars where dollars are never introduced in circulation.

In every particular application (a specific activity), we represent a space of a positive integer dimension just listing its dimensions in a definite order. This order may be of practical importance, or may not be. This sequencing does not change the space itself, which implies the entire ensemble of the possible representations, without an absolutely preferable ordering. Still the collection of choices is in no way arbitrary; it is exactly the common basis for all the possible representations of the space that we call its (geometrical) dimension. In other words, dimension is understood as a hierarchy, producing multiple hierarchical structures (the positions of hierarchy). For example, one can observe that the Cartesian product of two spaces with (different) dimensions N_1 and N_2, is obviously non-commutative, though the overall dimension will equal $N_1 + N_2$ in any case. In this model, each many-dimensional space manifests itself as a variety of the decompositions of the total dimension into the sums of partial fragments, which can be graphically pictured as a number of tree-like structures (the possible unfoldings):



plus all the permutations in the sequence (a, b, c). In real life, some variants may be practically unfeasible. Thus, to get into an apartment in a city house, we need to first get in, and then make use of an elevator (or a staircase); the inverse order would require the art of climbing the walls.

2. Constraint

The notion of constraint is widely used in analytical mechanics. It can readily be associated with negative dimensions. Indeed, while an additional dimension adds a degree of freedom and increases the total dimensionality of the space, a constraint, conversely, blocks motion along a certain line (not necessarily straight) thus effectively diminishing the dimension of the problem. The simplest constraint must therefore be treated as a space of dimension –1. Any combination of constraints will produce a constraint of a higher rank, producing space of any negative dimension.

The way of imposing a constraint depends on the space where it is to be defined and the choice of parametrization. In particular, when a space is represented by some coordinate system, a constraint can be expressed by an equation somehow combining the coordinates. However, like with the positive dimensions, constraints do not depend on such specific parametrizations. While a dimension conveys the idea of an objective relations between things, a constraint refers to some relation between such relations; this is, so to say, an ideality of a higher level. Still, in many practical cases, when we are primarily occupied with the fundamental contrast of the material and the ideal, rather than the detailed structure of ideality, the distinction between dimensions and constraints is formally irrelevant, and one is free to combine them in any order to produce all kinds of spaces.

Obviously, the spaces of the same overall dimension can be structured in many ways, in accordance with the mode of adding dimensions and imposing constraints. Some combinations may be impossible to practically implement. In an abstract theory, assuming the formal acceptability of all such constructions, the total dimension of a space with constraints is a general characteristic of the possible unfoldings (positions) of hierarchy, hierarchical structures. For example, in atomic physics, a theory of the collective motion of and atomic electron and a hole will make a three-particle problem (accounting for the field of the atomic core); however, the atom is neutral as a whole, and its complex structure will only manifest itself at a closer contact.

3. Projection

Just like constraints, projections effectively diminish the dimension of a space, but they do it in a different manner. A projection relates an N-dimensional space to another space of the dimension Q (the component space), which can be treated as internal space contained in every point of the original space. The dimensions of such inner space are called projections; their dimensionality is evaluated as N / Q. In particular, the components of a one-dimensional space have the dimension of $1 / Q$.

This allows constructing spaces of any rational dimension. Imposing constraint on projections rather than the dimensions of the original space, we obtain spaces of negative rational dimension; a projection of an elementary constraint will then have the dimension of $-1 / Q$. One could readily observe the kinship of negative constraints to the common orthogonalization procedures; thus, projecting a “vector” onto the inner dimensions that are orthogonal to it, we get zero.

A point of the original space can be reconstructed by a complete set of its projections. In the inner space, this means constructing a space of Q dimensions from individual inner dimensions. In terms of outer (“Cartesian”) products, we get the usual equality

$$\left(\mathbf{R}^{N/Q}\right)^{Q} = \mathbf{R}^{N} .$$

Of course, one is to accurately account for the possible interdependencies, to establish a kind of “orthogonality”, which may only be locally reachable in nonlinear dynamics. This does not change anything in principle.

Real numbers are commonly defined as classes of converging sequences of rational numbers, or the sections of the rational set. Following the same logic, sequences (hierarchical structures) of spaces of rational dimensionality will produce real dimensions. It is important that it is the geometrical (natural) dimension of the space that is real, and not an outer (topological) dimension. In general, no topology is implied by geometry, and geometry does not depend on topology. Some special theories may correlate the procedures of fractal construction to rational-dimensioned spaces, so that topological dimension could follow from geometry, with a kind of conceptual isomorphism. Still, let us stress once again, isomorphism is not equality. For instance, the point of the segment (0, 1) can be neatly mapped to the points of the segment (1, 2), with the entire structure preserved; this in no way means that $x = x + 1$.

4. Index

Indexing can be understood as the opposite of projection: instead of unfolding an inner space of every point, we attach some outer object to it, thus effectively increasing the overall dimension. This outer thing is used as the “name” of the point, its formal label that can change with the transition from one index system (frame of reference) to another. Such names can be of any nature at all, not necessarily from the mathematical domain. Take, for example physical fields or toponyms. In mathematics, however, indexed spaces are quite common as well. For instance, any coordinate system is of that very kind: we label a spatial point with a cortege of numbers reproducing the chosen sequence of the dimensions of the space. This is an elementary index space of the dimensionality 1, a “vector”.

Alternatively, in each point, we can construct a matrix (a tensor of the rank 2) rather than a vector. The components of the tensor we mark with two indexes, so that, if the substrate space has the dimension of *N*, the components of the tensor will form a space of the dimension N^2. Obviously, indexing with *k* indexes corresponds to the power *k* of the dimension of the original (configuration) space.

One might think that the power of a number could be naturally introduced as repeated multiplication:

$$N^k = N \cdot \ldots \cdot N \quad (k \text{ times}).$$

For the square of a space dimension, a similar approach would seemingly give

$$\mathbf{R}^{N^2} = \left(\mathbf{R}^N\right)^N \sim \mathbf{R}^N \times \ldots \times \mathbf{R}^N \quad (N \text{ times}),$$

and one could fancy longer chains like that. The problem is that the dots in such expressions do not denote an elementary operation; in fact, this is a sort of “quantifier” which belongs to the next level of logic and hence cannot be defined in terms of the original object area. In fact, we thus mean some activity in the base space. This process may sometimes be programmed, to a certain extent. Much more often, an informal procedure is implied, which makes it an inexhaustible source of ever new mathematical structures. Noting that such repetition, in general, does not need to be limited to an integer count (since we are going to discuss spaces of any real dimension), the above “trivial” definition is utterly unsatisfactory; that is why we accept from the very beginning that exponentiation is an operation of a special kind that cannot, in general, be reduced to multiplication. Still, for a small integer number of indexes, some indexing systems allow establishing a correspondence (isomorphism) between the spaces produced in alternative ways, to preserve the “correspondence principle”.

Dimensional indexing will naturally reproduce the usual properties of the power:

$$1^k = 1, \; N^1 = N .$$

Indeed, if each index may only take a single value, an object with any number of indexes will have a single component; when there is only one index, the number of components equals the dimension of the base space.

Any index lists the dimensions of the base space in a definite order. As mentioned before, this corresponds to unfolding the hierarchy of the space into a specific position. The same hold when the space is being constructed with constraints and projections. The sequence of “constructors” plays the role of a spatial dimension in respect to the index set. Of course, such a space admits index constraints and inner dimensions of the indexes. That is, the number of indexes (and the components of the index space) is generally expressed by a real number. Thus an arbitrary real power of dimension is defined.

The index space of the dimension 0 is readily associated with a scalar field, a numeric function on the base space. Obviously, any dimension in the zeroth power will give zero. For definiteness, let us accept that any power of a zero-dimensional space is to produce an index space with no components. Here, however, there are alternative possibilities: for instance, one might prefer construction of the molds for indexed objects, allowing a definite index structure, but without actual components in the possible positions; such an abstract object does not refer to anything and has nothing to do with the structure of the base space. This is quite like distinguishing complex numbers with zero (or infinite) modulus and a range of phase values (which in not a common choice in the present mathematical theories).

The power (–1) of a dimension *N* is a constraint of rank *N* in the index space, which is equivalent to the space with the negative dimension (–*N*). In the tensor model, such a constraint could be associated with the lower (covariant) index, in contrast to the upper (contravariant) indexes for the positive dimension. Imposing this constraint is simply a convolution of the constraint with one of the upper indexes, so that the total number of indexes will be diminished by one, as one could intuitively expect. However, there are other types of constraint that cannot be directly related to a power of some base space. Thus, merely fixing the value of a single component (or a combination of the components) of a power object, we get a constraint of the dimension –1 on the index space. In general, the values of multiple components get thus interrelated; in respect to the base space, such constraints become *symmetries*. They do not change the dimensionality of the problem, while significantly influencing dynamics (once again, mind the difference between the geometrical and topological dimension). When there are too many such constraints (above the dimension of the space), symmetries become constraints.

The square of a constraint (a space with the dimension –1), by its sense, is a constraint imposed on a constraint. This effectively corresponds to unfreezing a degree of freedom. That is, for spatial dimensions, $(-1)^2 = 1$.

5. Branching

Indexing (exponentiation of dimension) is a transition from one level of hierarchy to another level, where the objects are structured unlike the objects (points) of the base space. To produce a regular geometry, we need a special operation enumerating of the components. Of course, such an ordering can be differently achieved. In principle, this does not differ from the enumeration of the dimensions of the base space; however, the necessity of "lifting" the procedure of exponentiation in its result, the transition from many indexes to simple succession (a higher-order index), is always present in the background; this is not a formal trick, but rather a practical act related to the choice of the object area. Given the presence of constraints, such a transition could be compared to canonical transforms in analytical mechanics.

In general,

$$\mathbf{R}^{(N^k)^{1/k}} \neq \mathbf{R}^N .$$

Each instance of exponentiation (albeit to a fractional power) moves us to a higher level of hierarchy, which cannot be unambiguously reduced to a lower level, since there are different unfoldings of the hierarchy, and the same higher level object may result from different hierarchical structures. The parent structures of a power space could be called the branches (folia, replicas) of the base space. These are higher-level objects, which have different dimension but still somehow correspond to each other (up to isomorphism). For example, a two-index space of the dimension 1 can be obtained as a square of either a one-dimensional space, or an elementary constraint; two branches are thus defined, each representing a specific position of the hierarchy of the index space. For index spaces of a more developed structure, the number of branches may increase, and even be infinite.

The branches of an elementary constraint on the index space of the rank 2 (that is, the restrictions on the components of a square matrix) are of a particular interest for the mathematics of dimensionality. Such a constraint has geometrical dimension –1, while the search for the branches means taking a square root. In this way, we come to the notion of imaginary dimension (+*i*) and imaginary constraint (–*i*), which can be further expanded into a theory of spaces of any complex dimension, in the simplest case, representable by the product of the real and imaginary parts.

Quantum Set Theory

The fundamental notions of the classical set theory are never formal: a set, an element, membership and absence, equality and difference… Depending on the conventional usage rules, different classical theories may emerge; in any case, the formal axiomatic carcass cannot be treated as a definition, but rather should be taken for a "constraint" (in the sense of theoretical physics) narrowing the range of the relevant constructs, while never eliminating terminological ambiguity. When it comes to drawing analogies from physics, we have to operate within a specific interpretation; this is a regular situation in any science and it does not pose any serious problems as long as no abstraction is deemed to be absolutely preferable and the natural scope of a particular science is always kept in the mind and respected. On the other hand, all the possible models are equally admissible, and one is free to expand whatever, however exotic, in the hope that the experience might come handy on an occasion.

Traditional mathematical objects are essentially static: they are just somehow given, and a mathematician is only to study the already available features. Well, this is exactly like we used to look at the world before the XX century, under the reign of classical physics. A set thus understood is an outer thing on itself insensitive to all kinds of manipulation. Somebody we don't know has made it for us that way, and we can count on that it won't change or disappear while we are still working on it. Accordingly, given several sets, we can cook them all at once and the result will be as eatable for anybody else.

For every set, the presence of elements is the principle #1. We do not mean that some other entities cannot have elements, they too; still, an entity without elements can be anything but definitely not a set. In particular, the phrase "empty set" is nothing but an abbreviated form of a statement like "there is no set such that…" In an appropriate context, this negative existence may acquire certain trait of an object (with a variety of equally admissible paradigms), but that won't in the least make it a set.

One set is not like another. When it is small, we can enumerate its elements one by one or grasp them all at a glance; in the worst, we can suggest an effective procedure for sorting them out in a final (within a current activity) time. For very big sets, this is no longer an option; the maximum we can hope for is to find an illustrative analogy: it is like a segment of a curve, a collection of functions *etc*. For some sets, we cannot afford even that; there is an opinion that such huge conglomerations should not be called sets proper. All right, let's pretend to have got such a monster, one way or another. Now, there are two problems: first, anything relatively well-formed we encounter in our life may be or not be an element of a given set; that is, there must be an effective procedure to determine the membership of anything at all to our (however large) set. Second, declaring it to be a set, we must be able to support our words explicitly presenting at least one of its elements to the public; in other words, we need a practical ability of plunging the hand into the set and getting out ("selecting") something that would undoubtedly be recognized as belonging to it (moreover, specifically as an element, not just a subset). The both procedures may come highly nontrivial; indeed, it is the technologies of the kind that any branch of classical physics is to eventually develop: every science is to historically grow to a clear recognition of its object area.

Any idea of membership in the classical set theory comes from explicit set construction, that is, given a well-defined object we only restrict (directly or through imposed constraints) the right of different object to belong to a given set. No comprehensive universe can ever be defined within set theory; any known attempts have always lead to a logical circularity, the premises exploiting the features of what is to be eventually obtained.

With sampling, things are no lighter. Admitting that we can be satisfied with a conventional technology, there is still a possibility of sets differently reacting to what we do: no arbitrariness, no mathematician's caprice, but rather the demand of the object area we mean when using that very kind of mathematics. The traditional set theory deals with collections that cannot contain an object of a given type more than once. When we draw out an element in the course of sampling, we obtain a different set that does no contain that very element. In other words, a set as an integral whole splits (decays) into two parts: an element and the residual set (absent in certain cases). We are well acquainted with such transformations in physics and chemistry. Of course, in that picture, human intervention in a mere locution, and we can as well consider interacting sets interchanging elements due to some objective happenings, in an entirely automated manner.

The complementary approach is to allow several elements of the same kind within a single aggregate, which should not probably be called a set, but rather a "bag". Provided there is an effective procedure for determining the number of identical elements, the bag will virtually be a set; in general, this is not the case. For set-based bag, one still has opposite choices. For instance, take a two-level structure, with the elements well-distinguishable on the lower level, but merged in equivalence classes on the upper level. On the contrary, in a statistical representation, we speak about the probability of an element's membership in a set: the number of identical elements is effectively divided by the total number of elements. For very large sets, the statistical approach may be preferable (and even the only possible). Any intermediate structuring would involve some generalized statistical weights and the corresponding statistical sums; practical considerations stand behind a particular choice.

With all the diversity of paradigms, classical set theory refers to some “accomplished” sets that can be studied at any convenient pace. For a classical observer, any structural change will look like a “singularity”, “catastrophe”, or “phase transition”: the end of a world, and the beginning of another. Here, we are interested in the smooth evolution within the zones of continuity; in their meeting points, any stable structures are assumed to completely form during the time (or instant) between the two consecutive acts of “measurement”.

A quantum experiment is primarily different from a classical setup by the observer’s intervention in the motion of the system to be observed: first, we prepare something observable, and then try to figure out what we have really prepared. Classical experimenting is following that scheme too, but the classical act of creation is in no way related to the process of observation: the two activities are well-separated in space, time, or however else; roughly, an already prepared system lives long enough to forget about those who gave birth to it well before somebody else would wish to study. A quantum system is consumed immediately, in the very time of its arrangement. Similarly, a movie differs from a stage show, correspondence from live talk. As usual, there are minute gradations, and the distinction of quantum and classical sets can only exist on a specific level of hierarchy unfolded in one of the possible directions.

Quantum dynamics proceeds entirely inside a classical singularity point; for a quantum description, the whole classical motion before and after restructuration will serve as the initial and final state, the asymptotic conditions. The matter of the fact is in there, but we cannot directly observe it (without breaking a quantum system into classical parts) and hence must guess by the side effects, comparing the incoming and outgoing structures.

A quantum set (or, generally speaking, a bag) is a formally prepared system in one of the possible states; this can be conventionally represented by the abstraction of a “state vector” $|A\rangle$. All sets that can be produced using the same preparation technique constitute a kind of universe: metaphorically, we call it a “configuration space”. To determine whether an element *a* belongs to a set *A*, we compute the number (an “amplitude”) $\langle a|A\rangle$, with the square of its modulus taken for the degree of the element’s membership in the set (statistical weight). Using the same vector metaphor, one could consider an element as a “functional” over the configuration space of a specific level. The ensemble of such functionals will determine the object area of the theory. In other words, this is what we want to eventually get in the course of activity, its practical outcome, a product.

A one-element set containing a single element *a* could be denoted as $|a\rangle$, with $\langle a|a\rangle = 1$ (in general, the character “1” may stand here for something far from being a number; for instance, a kind of δ-function, that is, yet another functional). For all the other members *b* of the object area, $\langle b|a\rangle = 0$.

So far, no significant difference from a classical set/bag theory have been introduced. The transition from probabilities to amplitudes does not change anything on itself; it is no more than a kind of substitution of variables, a change of viewpoint; with the classical weights of the elements as the only outcome, the benefits of the new formalism are rather obscure (if not doubtful). This will be so as long as we deal with the earlier prepared sets and do nothing except measuring the degrees of membership. Any kinematic picture is bound to expand on the level of macroscopic (classical) observer, since all we need from a science is a number of practical things ready to use in our everyday life. Essential differences can only be found on the level of dynamics: with classical sets, we combine the observables (statistical weights), while the interaction of quantum sets means combining amplitudes.

As we discuss mathematics, which is basically a science about abstract structures, dynamics cannot directly enter a mathematical theory and it must be represented by specific structures. Within the quantum paradigm, we associate any state change (motion in the configuration space) with “operators”.

Immediately, this concerns set transformations, that is, set operations defined on the current universe; however, any set comparison is also a kind of transformation, and hence various relations between sets must also be representable with operators, though possibly of a different kind.

In particular, the relation of an element's membership in a set requires reconsideration. The idea is blazingly simple: one cannot compare qualitatively different things (belonging to different levels of hierarchy). Elements are comparable with elements, sets with sets. To compare elements with sets, we need to somehow bring them to the same type. The traditional notation $a \in A$ is a mere abbreviation for a sequence of nontrivial acts, each with their own conditions of feasibility. Most such assumptions never come to a clear wording: usually, a mathematician just believes that his abstract world is regular enough to justify any formal manipulations that lead to the desired result. Numerous logical strains drive some mathematicians to abandoning the very notion of an element, so that the whole theory is restricted to set comparison; this does not help much, just postponing the difficult questions that will come back elsewhere, in a new formulation.

In quantum set theory, sets are represented by "state vectors", while elements are represented by "functionals". The difference strikes the eye. Establishing any interrelations is quite an undertaking, with different technologies leading to very unalike theories. Still, in any case, we have two basic options: either elements are to be transforms to sets, or the other way round, sets to elements. The third way, bringing the opposites to a new synthetic entity, would virtually break the boundaries of set theory proper.

The former approach is possible using a special set operation, projection: its intuitive sense is to pick out a part of a set (or a subset, in the language of the traditional set theory). For a single element *a*, the corresponding projection operator is commonly written as $|a\rangle\langle a|$, so that the projected set (the outcome of projection) would take be pictured as $|a\rangle\langle a|A\rangle$, which apparently puts each element (functional) in correspondence with an appropriate one-element set (vector). So, in a given object area, a set formally becomes a linear combination of one-element sets:

$$|A\rangle = \sum |a\rangle\langle a|A\rangle = \sum |a\rangle \psi_a(A) .$$

The same set can be considered in a different respect (in another object area), which would results in a new expansion of the same type:

$$|A\rangle = \sum |b\rangle\langle b|A\rangle = \sum |b\rangle \psi_b(A) .$$

This "completeness condition" is often formally formulated as

$$\sum |a\rangle\langle a| = 1 , \ \sum |b\rangle\langle b| = 1 ,$$

but we must keep in the mind that the configuration spaces for elements *a* and *b* need not coincide: in general, no transition from one "basis" to another is meant, as we can treat the same thing many alternative (or complementary) ways. For instance, a graph can be represented by a collection of nodes connected by arrows; but it can also be treated as a number of arrows connected by nodes. One can control a computer using a keyboard, in a command-line interface; but the same controls are often available in a graphical interface, with just a mouse click. In both cases the effect is the same, despite all apparent differences.

It should be noted that the dimensionality of a basis is not related to the size of the set. For example, in atomic physics, the same quantities can be evaluated by either an integral over the continuum states or a sum over a specially designed discrete basis. In logic, we use a two-element basis (just **T** and **F**) for a whole lot of the possible statements; their object content and practical meaning are irrelevant for logical valuation. Everybody knows, that the same problem can be solved using a standard but cumbrous approach, or in an unexpected and elegant manner.

Generally speaking, the product of an activity is different from its object (the raw materials and available technologies). Still, in certain situations, both the object and the product are considered in one of the possible aspect, so that the difference is effectively lifted. Thus, in market economy, both material and spiritual reproduction are regarded as the metamorphosis of exchange value; similarly, in the structure of a scientific theory, deduction moves from a number of truths to other truths.

In real life, an orthogonal basis is often more comfortable and illustrative; still, just like in classical theory, orthogonality is not indispensable: the presence of one element in a set may (to certain extent) mean the presence of another. In quantum theory, orthogonality means that every one-element set is a eigenstate of a specific operator.

The transition from one basis to another can be expressed as

$$|A\rangle = \sum |a\rangle\langle a|A\rangle = \sum |b\rangle\langle b|\rho|a\rangle\langle a|A\rangle .$$

That is,

$$\psi_b(A) = \sum \rho_{ba}\psi_a(A) .$$

In the simplest case, when the basis spaces of *a* and *b* are defined in the same universe, the “density operator” ρ can become identity, and we speak about equivalent element representations of the set. In general, one still needs to bring one object area to another to ensure comparability; the way of such reduction depends on the intended applications.

Treating sets as collections of elements implies the ability of explicit construction. As in classical theory construction is well separated from observation, there is an illusion of the simultaneous presence of all the elements of the set: they all are in the view field, and no element can be preferred. Quantum set theory represents addition or removal of an element by the corresponding operators: the notation $a^+|A\rangle = |a;A\rangle$ means that, acting with the “creation operator” a^+ onto the set *A*, we obtain a new set that is likely to (but, as indicated below, will not necessarily) contain the element *a*; we expect that $\langle a|a;A\rangle \neq 0$. The inverse operation is to act with the “annihilation operator” a^- onto the set $|a;A\rangle$, admittedly restoring the set *A*. Any finite extensions of a given set *A* can be constructed in this manner; here, the set *A* plays the role of “vacuum”, a reference state for the rest of the theory. One might get tempted to take the empty set for the base and thus develop an “absolute” theory. However, the empty set is not really a set, and we should not treat it as a regular set, and, in particular, we cannot act on it with any operators defined for real sets. Similarly, in modern physics, vacuum is a sheer conventionality, a level of reference. Coming across a formula like $a^-|0\rangle = 0$, we must take it in the idiomatic sense, as an expression of the a variety of constraints imposed on the physical system; in many cases, annihilation operator for a particle *a* can be considered as creation operator for its antiparticle: $a^-|0\rangle = |\overline{a}\rangle$, so that a system might admit states with both particles and antiparticles (like an electron-positron pair, or the coupled motion of the free electron and the ionic hole in atomic ionization). Nothing prevents us from considering $a^-|A\rangle = |\overline{a};A\rangle$ as a set with a hole (an anti-element); the specific implementation of element addition or removal depends on the intended applications. For instance, adding an element to a set will not necessarily be a kind of creation: it may just increase the numbers of elements of the kind (like in the case of electron capture by an atom or an ion). In the same way, element annihilation may just diminish the “weight” of that element in a set, provided the element and the hole instantly annihilate leaving no trace of the event in the resulting set. In the simplest case, when the set is not allowed to contain more than one element of the same kind, creation and annihilation operators are idempotent: $a^+a^+ = a^+$, $a^-a^- = a^-$.

A sequential application of several creation/annihilation operators will produce many-element sets:

$$a^-c^+b^+a^+|A\rangle = |\overline{a},c,b,a;A\rangle .$$

The order of operation may be quite significant, and it is only in very special cases that the set $\{\overline{a},c,b,a\}$ can be identified with the set $\{c,b\}$.

With quantum elements “interfering” inside a set (however this interference is implemented), the state $|a; A\rangle$ can no longer be understood as $|a\rangle|A\rangle$. Some very simple sets can be formed as products of one-element sets (the eigenstates of a particular operator); such states (and their non-degenerate combinations) are called "pure". Given a "complete" basis, we can "enumerate" the elements of the incident set:

$$|A\rangle = \sum |z\rangle\langle z|A\rangle,$$

so that

$$|a; A\rangle = a^{+}|A\rangle = \sum |b\rangle\langle b|\rho a^{+}|z\rangle\langle z|A\rangle = \sum |b\rangle\rho_{bc}\langle c|a; z\rangle\psi_{z}(A).$$

In this way, addition of an element to a general set can be reduced to a union of two-element sets: the new element *a* is to be sequentially coupled with each of the members of the base set. This obviously corresponds to the similar expansion in the traditional set theory:

$$\{a\} \cup A = \bigcup_{c \in A} \{a, c\},$$

though quantum theory also demands accounting for interfering modes of virtual transition from one state to another.

Since we relate creation and annihilation operators to elements rather than sets, the union of two arbitrary sets is not always definable. Still, when the two sets have been produced starting from the same incident set (the "base", or "vacuum"), there is an option of considering the composition of the corresponding production operators as production of the union. This might be compared to the atomic states with different degrees of ionization. For yet another analogy, take the production of natural numbers with the only fundamental operation, the increment; the sum of two natural numbers is already an expansion of the original theory: this binary operation is imported from outside, being defined on a different level as a class of isomorphisms.

In certain cases, it is possible to define the union for the sets produced from difference base sets; this corresponds to the transition form atomic to molecular physics, with only the "valent" electrons and holes participating in the formation of the whole, while the atomic (ionic) cores are treated as relatively independent of each other (that is, the vacuum for the union equals the product of the original base states). Numerous options are possible, here too. Thus, one could compare the classical union as an analog of the covalent bond, with all the electrons equally belonging to each of the atoms in the molecule. In the opposite case, we obtain something like ionic bond, when the elements of one set get compensated by the holes in another. There is also an analog of the hydrogen bond, with no real union, but rather an "artefact" of the usage of a common basis.

The generalization of finite sets expansions onto infinite (countable or not) constructs is rather straightforward. From the practical viewpoint, it means considering a higher level, reflexive activity: instead of performing individual operations, we start constructing those operations using a regular approach. Like any hierarchy, set production can be folded into a "point", and then unfolded into a different hierarchical structure.

All the possible "interactions" between sets (set-theoretic operations and relations) are expressible through the combinations of creation and annihilation operators. Every special theory involves a specific collection of fundamental (elementary) interactions, so that the rest of the theory could be deduced from this axiomatic core. Since we are to eventually obtain a quite definite product, we reduce the result of each operation to the same reference basis; that is, the different combinations of elementary operations (the sequences of interactions) may lead to the same (in the sense of practical indistinguishability) set, while the interference of such virtual processes is reflected in the specific overall structure (spectrum) of the resulting set, up to the impossibility to obtain certain sets from the base (which is known in physics as selection rules).

This is the high time to ponder a little on the meaning of set comparison. What does in really mean, "to be the same"? The equality of elements is an entirely practical issue, it is determined by the organization of the object area. As for the equality of sets, opinions differ. It is usually said that two finite sets are equal if, and only if they contain the same elements. However, even that intuitively appealing definition is implicitly based on very thick assumptions about the object area and the sampling procedure (starting from the very possibility of enumeration). The situation is much worse with infinite sets, where we need to enumerate the elements of the both sets and compare them to each other in a finite (or even infinitesimal) time. Once again, quantum physics readily comes to mind, with the finer details of interaction "packed" in a single macroscopic point (or a moment of time), with the only observable outcome of a statistical distribution, a spectrum. Still, a similar sheme is possible in classical theory as well: consider one of the sets as a kind of a filter, a barrier, with the incident flow of the other set's elements that can be absorbed (or reflected) by the similar elements of the filter set; if there is nothing on the other side of the barrier (no outcoming elements), one can state that the incident set is less than the filter set (hence being its subset). Reverting the situation, with the incident set and the filter interchanged, we test the inverse relation: is there is still no output, the two sets are equal.

Obviously, this mental experiment is only one of the possibilities; however, it is enough to comprehend the idea of set comparison as a synopsis of numerous assumption about the character of interaction and the principles of dynamics. Just change the experimental setup, and you may get an entirely different picture. Well, there is nothing really new: for instance, there are different mathematical definitions of dimension, and we need to investigate the range of their compatibility. Similarly, with very large sets, we speak about their equinumerosity (or, at best, isomorphism) rather than true equality. Still, every man of reason would perfectly distinguish the sounds of speech from the graphic signs denoting them in the international phonetic alphabet: one can never convert one into another; the two sets are interrelated but not equal. In the same manner, even natural numbers are not the same as odd numbers, despite the fact that the two sets can be entirely mapped onto each other. The same score can be played with a violin, a piano, or an organ; but these instruments are in no way "isomorphic" in an orchestra.

The quantum paradigm brings in certain amendments, adding a "built-in" uncertainty, "partial" membership. With all that, the procedure of "filtering" one set with another is perfectly reproducible in quantum set theory; moreover, the transformation of a set into a filter here becomes a simple formal trick: all the element creation operators for one of the set must be replaced with the corresponding annihilation operators. As the resulting "holes" ("anti-elements") annihilate with the elements of the incident set, we immediately get the spectrum of differences in the end. That is, for the sets

$$b^+ a^+ \left| A \right\rangle = \left| b, a; A \right\rangle$$

$$y^+ x^+ \left| Z \right\rangle = \left| y, x; Z \right\rangle$$

the result of their comparison is given by the amplitude

$$\left\langle y, x; Z \middle| b, a; A \right\rangle = \left\langle Z \middle| x^- y^- b^+ a^+ \middle| A \right\rangle = \sum \left\langle Z \middle| \nu \right\rangle \left\langle \nu \middle| x^- y^- b^+ a^+ \middle| \mu \right\rangle \left\langle \mu \middle| A \right\rangle = \sum \bar{\psi}_\nu (Z) D_{\nu\mu} \psi_\mu (A) \, .$$

Under certain conditions, given the equality of elements $a = x$, $b = y$, this expression will evaluate to $\left\langle Z \middle| A \right\rangle$, which is unity for equal base sets. Of course, in more complex constructions, such reduction to unity will not guarantee the complete equality of sets; however, if the virtual transitions can compensate each other to that extent, this means that one of the sets cab be effectively transformed into another, and hence be its fair replacement in a practical sense. Considering this expression as a function of some "macroscopic" parameters, one gets a full-fledged spectrum, where any virtual compensations will reveal themselves as structural peculiarities (*e.g.* resonances).

Quantum set theory does not just extend classical theory; it eventually suggests a bunch of specific implementations suitable for particular purposes. Similarly, in physics, any "theories of everything" admit numerous observable "landscapes". The choice is never arbitrary; our practical needs will select the acceptable solutions. Such practically-oriented mathematics will no longer be a mere play of thought: it will become sensible and truly meaningful.

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Abstract Pictures

The science of mathematics (as any other science) is to reduce our daily problem to sheer formalities. And that's alright. Since, otherwise, we would get stuck in reinventing whatever has already been experienced, with no time for far-reaching creativity. However, as long as a human being is different from a bee, one cannot be completely satisfied with the efficient ways of being; there is an ardent desire to grasp the whole thing from elsewhere, from a neighboring universe. Which is commonly known as intuition. Just cultivate it, and the formalities are no longer restrictive, and no distance is too far to march.

Well, there are people and people. Intuition on one kind may be inapplicable to a person of another constitution. Some will enjoy the idea of a sequence of actions: the "Dao", so to say, the procedurality as such. With the funny pictures as an occasional result. Some others, on the contrary, would hate being too algorithmic: they prefer a visible thing, to look at it from every side and finally decide. Similarly, in music, there are both melody seekers and harmony feelers. The third way is related to instrumental intuition, practical sense, the ability of build it, picture it, and play it on the fly. Which is commonly known as talent.

Mathematicians have intuition of a special flavor. Any order is due to numbers; any picture are in the reign of geometry. Still, numbers tend to gradually turn into abstract structures, while geometry gets drowned in homeomorphisms and reduced to numerical (topological) invariants. The talent of math thus grows into logic, the sense of the essentials, the ability to withdraw from the particulars.

In real life, particularity is much more helpful. It's a pleasure, to discover still finer details, one by one. With all that, any subtleties are chaotic and dull on themselves; that is why we need to attach them to some solid principle, thus putting the ocean between firm coastlines that would justify any pain and trouble. With a bright and vivid abstraction, the storms are tender, and no calm is dead.

Alright, let us amuse ourselves a bit with browsing the fancy of visualized numbers.

Numbers are all different from each other, just like people. Quite naturally, everything begins with counting: one, two, three, four, five, I caught a fish alive; with a fish live indeed, I don't need to proceed… Why? Because the basic idea of the process is already clear: there is a certain direction to mark-up at some pace.

In the same manner, rational numbers are readily visualized as a couple of independent directions, with a specific pace along each. Something like a planar grid. Now, a practical fish breaks in and asks: what if we are going like that, but in the same direction? Will we come to anything common, or never? This happens to depend on the step size: for some combinations, there is a consensus, while some others give no chance. In the first case we speak of commensurability, or rationality. If everybody was rational, the common order would be easily attainable, so that all numbers were pictures by the points on the same line, and one could always find whether a particular quantity was great enough, or lacked something. Is that any different from natural numbers? Yes, it is, since we have to seek for a common measure every now and then, and the same thing will be called differently, depending on the unit system.

Perfect. Let everybody come to an accord and be happy. Unfortunately, not everybody can. There are all kinds of incommensurable, irrational… Just like in real life: they are much more plentiful than the rational elite! They are tremendously numerous, and this is not a mass to ignore. They form the very substance, the reality of the line, where rational numbers are so comfortable and spacious; that is why they are called real numbers. For each rational, there are more irrationals around than the quantity of rational numbers altogether. In any however tiny neighborhood. On the other hand, every real number can be gripped in a vice of rational numbers, as tightly as needed; this conveniently restores the complete linear ordering.

In the world of limited opportunities, there are no other completely ordered number systems. That is, any order at all brings the same image to the inner sight: a smooth line, which can easily be pictured as straight, provided we can close the eyes and leisurely glide along. This, too, has a name: continuity. Pits and bumps may break the spell sometimes; however, if they are not too many, the overall impression remains intact. After all, real numbers become a universal measure, with any distance expressible as a real number. Somewhere in infinity, beyond the borders of our world, there are other number-like things. For instance, the (ordinal) number ω (or rather ω_0) that is greater than any other integer. The number ω_1 is uncountably greater, and an infinity of (hyper) integer numbers can fit between the two. The overall order is thus extended onto infinite numbers; and for each infinity, there is the next, with no end. A serious person would not run after such intractable objectives; it is always commendable to keep in strict bounds and pursue a real purpose. That is, instead of idle promenades, better go straight from the point *A* to the point *B*. Or, in the numerical language, from zero to unity. Of all the real line, we are only interested in the segment [0, 1]. Thus we can be sure to reach the intended destination after a finite number of steps, for an arbitrary step size. To avoid racing through, just put a wall in the end. That is, we consider the segment [0, 1] together with its boundaries; some may wish to call it a closed manifold.

One is lucky to find that the number of points in the segment [0, 1] is no less than on the whole real line, and we do not miss anything. Rational number are also present here in full. How many? One cannot tell for sure; but this does not prevent us from naming. Let us denote the number of rational numbers with the term “aleph-zero” ($\aleph_0$), while the quantity of real numbers is associated with the word “continuum” (also known as “aleph-one”, $\aleph_1$). There are all indications that the “cardinality” aleph-zero is strictly less than continuum. There are different opinions on whatever lying in-between; theoretically, we are free to fancy an entity like that, but nobody has managed to show up with anything palpable. Just console yourself with the well-established existence of quantities greater than continuum.

Indeed, imagine that every point of the segment [0, 1] is to be labeled with a number from the same range. Obviously, this is always possible, since the number of labels is exactly the same as the number of point to label. In this case, we say that there is a function form the segment (the “domain” of the function) into itself. In general, some points may bear the same label; such functions are not invertible, as one cannot unambiguously identify the prototype for any given function value. Well, nothing special, this is how life goes. Still, we can take all the namesakes together and declare that they constitute a kind of community, a subset of the segment [0, 1]. Now, the collection of all such subsets is found to be much vaster than the real axis; this means that the numbers of all bounded functions on the segment is greater than continuum. There are even greater cardinalities. But why? To start with, some vision of just one greater-than-real thing would be quite a deed.

As a first impulse, why not reproduce the trick with projecting the plane onto the real axis, just like we did to produce rational numbers? But look, thus “closing the fan” we got no increase of cardinality: the quantity of natural and rational numbers are the same (aleph-zero). Similarly, the number of points in a plane is the same as the number of points on the real axis (continuum). Still the basic line of though is quite acceptable; we only need to follow it a little farther.

One can easily imagine a square or a cube. The generalization to higher dimensionalities is rather straightforward. Many usual statements about our “domestic” space are directly applicable to abstract higher-dimensionality spaces. Yes, extrapolations may occasionally fail; the search for such catch-tricks (and striking teasers) has always been a well-gratifying mathematical amusement. With all that, there is an intuitive idea of a many-dimensional space, a kind of an abstract picture. We know that an ordered pair of points (at least close enough to each other) determines a spatial vector, which has both length and direction. Length is just a number, and we can naturally evoke the picture of a line. Imagination does not support higher-dimension angles; still, any two vectors lie in a plane, and the plane angle between them can be extracted from the scalar product. With a little more effort, one can picture all kinds of bodies, of three or more dimensions. Something floating inside a hypercube.

The axes of a many-dimensional space can be enumerated in some specific order. The transition from one enumeration to another does not change the overall geometry; still, some of its properties may be more tractable in a dedicated representation. The sequence of dimensions determines the orientation of the space. As the very possibility of axis enumeration implies an outer observer taking the space as a whole, it is not always possible to transform one geometric object into another by a continuous movement within the space, preserving the orientation of the both.

Now, let's look at the space of functions from the segment [0, 1] into itself. Admit that every point x of the segment corresponds to an axis of many-dimensional space; the value of the function in the point x is treated as the coordinate in this dimension. Then, every function can be geometrically represented as a point in the hypercube with a huge number of dimensions. Yes, their number is continuum. But this does not deny the spatial essence of a function, and we still have the geometry of plane and a 3-dimensional body, with the rest built on by analogy.

Note that physics has long since dealt with the spaces of infinite dimensionality. In quantum mechanics, state vectors may have infinitely many components, with a liberal mixture of the discrete and continuous spectrum, often adding some greater cardinalities. This does not prevent us from doing sensible calculations (though, possibly, not too rigorous from the mathematical standpoint). In fact, physical spaces are even cooler: there are also inner ("spinor") dimensions in each point! Here, let us keep on with a rather modest imaging.

A source of infinite amusement is to determine the classes of functions that correspond to the typical geometric objects, like point sets ("crystals"), curves, planes, bodies. For instance, the main diagonal of the hypercube pictures the family of functions that are constant on [0, 1]. It is obvious that all the subsets of the segment [0, 1] lie in the vertices of the hypercube. Indeed, every subset is associated with a characteristic function that takes only two values: 0 or 1. A continuum-sized sequence of zeros and unities, by construction, specifies one of the vertex points. In particular, with the "natural" orientation of the hypercube, when the sequence of axes coincides with the segment [0, 1], the empty set is logically found in the origin of the coordinate system; it is represented by the continuous sequence of zeros. The opposite (the most distant) vertex of the hypercube corresponds to the sequence of unities representing the whole segment. Similarly, some other families of functions can be visualized as many-dimensional entities embedded in the hypercube. This may suggest useful implications.

Geometry will only respond to its name, when we can measure something in a space. In the ordinary (Euclidean) spaces, this is achieved using a multidimensional version of the Pythagorean theorem. By analogy, one can define the norm of a function as the distance from the origin:

$$\| f \|^2 = \int [f(x)]^2 dx$$

In the same manner, the distance between two functions is defined as the length of the difference vector:

$$\Delta^2 = \int \left(f_1(x) - f_2(x) \right)^2 dx$$

In particular, the distance between the opposite vertices of the hypercube equals 1, while the distance between any two «adjacent» vertices (forming a finite- or countable-dimensional hypercube) equals 0. This is naturally complemented with the notion of the angle between two functions:

$$\cos\theta = \int f_1(x) f_2(x) dx \Big/ \| f_1 \| \| f_2 \|$$

We do not need to precise the sense of integration in the above formulas. Each definition has a peculiar (and possibly useful) geometrical interpretation.

With this definition, certain thing look quite naturally. Thus, the norm of a constant function is trivially equal to its value. The distance between the functions $f(x) = a$ and $f(x) = b$ is $|a - b|$, while the cosine of the angle between them is always unity: as expected, such functions are parallel. The squared norms of the functions $f(x) = x$ and $f(x) = 1 - x$ are both equal to $1/3$, as well as the distance between them; the cosine being estimated as $1/2$, we get the angle of 60º. These two function represent the family of unitary transmutations on the segment [0, 1] (changing the orientation of the coordinate system). Clearly, such a tangled function can always be ordered by the function value and thus reduced to the same simple form $f(x) = x$; consequently, the norm of any transmutation is also $1/3$, while the mutual distances and angles may significantly differ.

This brings us back to the possible specifications of the integral. For one possibility, to compute the integral of any function, we first rearrange the dimensions of our hypercube to monotonically order the function values; after that, the integral is defined as the area under the resulting curve. Since only regular functions are involved, there are no serious technical intricacies, and any integral will evaluate in a number between 0 and 1. Of course, some precautions are still necessary. Thus, for a non-bijective (but otherwise smooth) function, the equal function values get to the "adjacent" points of the rearranged segment, and the corresponding measure (the elementary length dx) should be multiplied respectively. For an alternative picture, one might redefine the "density" of the points on the segment. However, this does not much hinder geometrical vision.

The characteristic functions of the subsets are, therewith, conveniently tractable. After rearrangement, every such function becomes a unit step function: zeros first, unity values to the end of the segment. The norm is then defined as the length of the unity part (which is exactly the number of elements in the set, its continuous measure). For example, the distance between two characteristic functions is determined as the measure of their union, minus the measure of the intersection.

Well, infinities are certain to have at least some crazy turns. For instance, the general formula evaluates the distance between the constant functions 0 and 1 (the opposite vertices of the hypercube) as 1. But everybody knows that length of the diagonal in the unit plane square equals $\sqrt{2}$! Here, it happens that the diagonal is unity, with all the edges of the same size. Quite a mystical thing.

At the second sight, it's no wonder. The purely Pythagorean diagonal of the N-dimensional unit cube equals $\sqrt{N}$. With $N \to \infty$, one obtains an intractable something of the $\sqrt{\infty}$ type. To normally work with such constructs, it is convenient to normalize all the lengths: in the finite-dimensional case, mere division by $\sqrt{N}$ is enough; for continuum, we get a kind of density in the sequence of the spatial dimensions. In the same manner, one could normalize the usual two-dimensional square too, which is equivalent to the choice of a different unit for its diagonal, so that to make its length equal 1, just like in the infinite-dimensional space. This is a quite logical approach: in physics we often switch to some "natural" unit sets; similarly, relativity relates all the speeds to the speed of light, which corresponds to some experimentally observed symmetries.

There is a much more annoying problem. The integral definition of the norm is not unambiguous, since any integral is evaluated up to an arbitrary contribution of zero measure. That is, in fact, we do not determine the distance between individual functions, but rather between some classes of functions. Mathematicians would not much bother: their science is entirely like that. A man-in-the-street would prefer a nicer definition of distance to comply with the visible separation of spatial points. For instance, a circle (or a sphere) with the center in an inner point corresponds to all functions equally distanced from a fixed function. The variations of zero measure heavily spoil the picture, since they form an infinite class of functions, and the corresponding hypercube points are everywhere dense. What, then, is left of the geometrical obviousness? Things are even more aggravated, when it comes to the metrical definition of the vicinity, neighborhood, open and closed classes of functions *etc*. Any topological constructs are no longer simple and intuitive.

Yes, one can observe that the class of all functions can never be completely ordered and thus packed into the real axis. Its cardinality is higher than continuum, and that does it. One could practice severe self-restriction, considering very simple functions (say, diffeomorphic to a constant). In this case the hypercube closeness will coincide with metric proximity, and one will operate with trajectories, smooth transitions in the function space. However, many interesting functions will be left beyond that narrow scope, including permutations and characteristic functions. There is yet another approach: instead of the common space for everything, consider a layered space, a hierarchy of function classes, so that the usual metric is retained within each class, while any aggregate estimates are feasible on demand. For example, an arbitrary function can be represented with a (direct) sum of components: finite point set based subsystems, infinite discrete subsystems, and a number of continuous (or piecewise-continuous) areas. Instead of a single distance, we thus get three respective distances, one for each independent component (level). That is, instead of bluntly ignoring the regions of zero measure, we account for their contributions in a special way, treating them as singularities. In a different formulation, one could speak of singular measures (of the δ-function type). Provided the corresponding subspaces are orthogonal, the overall distance could be naturally defined by the sum of squared partial distances (normalized in a standard manner). One is free to suggest any other modes of aggregation, depending on the practical needs. In this way, the functional geometry won't become too cumbrous, leaving much room for graphical intuition.

May 1992

Objective Set Theory

The traditional set theory says nothing bout the elements of a set. Admittedly, sets can become elements of other sets, without losing any of its set qualifiers while treated as an element. Such an approach suffers from too much generality, and the indeterminacy of the basic notions may lead to all kinds of paradoxes. To overcome this difficulty, mathematicians have already weakened the original universality and come to considering classes as different from sets, and less restrictive. The next logical step would also introduce the qualitative difference of a set and an element. One cannot substitute elements for sets or sets for elements; mixing elements and sets within the same argument (or formula) would be a logical fallacy.

In real life, any science is only applicable within its object area. Its abstract instrumentation will only mimic the real organization of the object. Otherwise, the theory is utterly nonsensical and of no practical use. Applied sciences do not need too much rigor: affordable recipes for everyday tasks are much more valuable. Even considering very different application, with some of them retaining a very high level of abstraction, we do not eliminate the obvious fact: there is something to study.

So, why not explicitly lay some object area in the basis of a theory? For mathematics, the particulars do not matter; it is only important, that there are somehow organized objects, and one cannot arbitrarily introduce imaginary entities. For a set theory, the object area provides some universe U, the ground level of the theory. On the next level, we get sets proper, as collections of objects from the universe U. In contrast to the traditional approach, sets cannot immediately belong to the universe: they are objects of a different kind. A one-element set $\{x\}$ is not the same as the element x. The relation of containment $a \in A$ or non-containment $a \notin A$ connects two adjacent levels of hierarchy. We can employ the usual notation for relatively small sets, just listing its element if the braces: $\{a, b, c, \ldots\}$. All thus included elements belong to the universe and cannot be sets. With this distinction, there are no problems of set formation by a common property: when the elements possess an objective property φ (compliant with the nature of the object and the logic of the theory), they can be taken together to form a regular set. The notation $\{x \mid \varphi(x)\}$ is acceptable for any objective properties; moreover, each set will determine some objective property as common for all its elements. When elements are clearly separated from sets, reflexive conditions like $x \in x$ are all out of reach.

Generally speaking, not all the collections of objects from the universe U will represent admissible sets. Different object areas may impose their specific restrictions. Without such constraints, for each individual object x from the universe U, there is a one-element set $\{x\}$. For a finite universe one could speak of the set containing all the objects from the universe; however, in general, such a "universal" set may be unavailable. For example, some objects may be not present in U all the time; also, some objects may have mutually incompatible properties and hence they cannot be contained in the same set. From the very beginning, we assume that every set contains at least one element; the common idea of an empty set $\varnothing = \{\}$ may only appear in an object set theory in a metaphorical sense, as an abbreviation for the phrase "there is no such set that…" We say that, by definition, set is something with elements. When there are no elements, this will refer to an entity of a different kind, to be studied separately.

If one set is equal to another ($A = B$), they are the same set, just differently labelled, as alternative construction paths may result in the very same collection of objects. The equality of sets is always understood as objective equality, the same property of real things.

One set can be a (proper) subset of another: $A \subset B$. Then we speak of a more specific property, narrowing B to its part A. Note that its is real objects from the universe U that are meant: some of them belong to one set without being contained in the other. Regardless of the number of elements (cardinality) the set B will be wider than the set A, if there are elements of B that do not belong to A, but not the other way round.

On the set level the usual definitions of set union $A \cup B$ and set intersection $A \cap B$ will hold, as well as the complement of one set to another $B \setminus A$. However, the formal construction does not tell us anything about the existence of the resulting set. Thus, the union of two sets may fail to exist in the presence of certain constraints, when some elements are incompatible. Since there is not empty set, the intersection of sets must contain at least one element; otherwise we honestly admit that the sets do not intersect (are disjoint, or disconnected). Similarly, the complement is only possible for a proper subset: $A \subset B$. In other words, the availability of any set-theoretical operations is determined by the nature of the object; conversely, we can judge on the structure of the universe, on the basis of the available set structures.

The collection of all the implementable objective sets can be treated as a higher-level universe, to construct the "sets" of sets; in respect to the object such constructs are classes rather than sets; that is classes are entities of a different level which they cannot contain objects from the universe and cannot be directly derived from the universe.

While set construction mainly refers to the objective properties, classes are more appropriate to convey the logic of the theory, the way of treatment and interpretation. The different theories of the same can be built using specific principles of class formation.

In the simplest case, when all the combinations of properties are possible, the class level exhibits a remarkable structure. For each one-element set, we can consider the collection of sets intersecting with that core set. In other words, we consider all properties compatible with a given object from the universe as a class, the collection of sets containing this very element. Since every set belongs to the class of any of its elements, the class level is entirely spanned with the above elementary classes. That is, classes can be uniformly mapped into the underlying universe, so that the level of hierarchy seemingly merge. This is known as hierarchical conversion: an element belongs to a set; but that set, in its turn, belongs to the corresponding elementary class. This three-level scheme represents here a formal theory, a mathematical model of an object area.

There is yet another direction of unfolding the hierarchy: any given set can be mad a universe (base) for the sets of the next level. Such sets could be called internal, as compared to classes ("external sets"). Internal sets obviously correspond to the subsets of the base; still the two levels cannot be identified, as elements cannot be directly compared to sets. One can also observe that the collection of internal sets is limited by the available subsets; this may serve as one of the possible definitions of a constraint. The classes of internal sets are constructed in the same manner. With the rich enough original universe, very complex hierarchical structures could be developed.

Now, let us take a couple of universes U_1 and U_2. Each of them will produce its own set-theoretical hierarchy. The components of these hierarchies cannot be immediately correlated, even at a similar level. This would be like arbitrary addition of millimeters with kilograms. The build a unified theory, we need to join the two universes in one, and then construct any higher levels. As usual, such a unification can be achieved in many ways. The two principal paradigms are given by sequential and parallel linkage, corresponding to time and space, the inner and the outer.

The sequential synthesis will produce a universe that is known as a (direct, or Cartesian) product of the original components: $U = U_1 \times U_2$. Each object from such a joint universe will be an ordered pair of original objects form U_1 and $U_2 : x = \langle x_1, x_2 \rangle$. This is important, which of the two component universes is taken as first, and which follows. Even in the case when the two universes coincide, they will enter the direct product each in its specific quality, as the first and the second. Any of these positions may imply its own constraints, and there may be oriented constraints limiting the number of the admissible pairs. The Cartesian-squared universe U^2 is different from mere numerical power; in particular, in each pair $\langle x, x\rangle$, the object x in the first position is different from the object x in the second: the same object is taken here in its different aspects. For example, it may be taken at different time moments, significantly changing from one to the next. Of course, formally, we might adopt the inverse order as well. However, this will result in a different formulation of the theory. Thus, instead of x_1 “precedes” x_2, we will employ a different terminology, saying that x_2 “follows” x_1. The synthesis of the universes is objective; it implies a practical usability of the joint universe, the real distinguishability of the positions in a sequence.

In this mode of synthesis, sets will contain various ordered pairs, accounting for the imposed constraints. Every element of a set includes the components of all the incident universes. Obviously, the existence of universal sets for both component universes would mean the possibility of treating the sets over the joint universe as subsets of the ordinary Cartesian product of sets. We know, however, that such global sets may do not necessarily exist, while the Cartesian products of the component sets do not always produce the same pair collection, especially in the presence of oriented constraints.

Sequential synthesis corresponds to considering different aspects in the same object, which are relatively independent and can sometimes be studied separately. For example, the temperature and pressure of a gas, the width of a river and its depth, the same society in different epochs. On the contrary, parallel synthesis joins one universe to another in an outer way, as two “parallel” realities (for instance, the disjoint areas of the same space, or the components and phase states of physical mixtures). In this case we speak of a (direct) sum, or a superposition: $U = U_1 + U_2$. This sum is commutative, as we are interested in the very presence of the object rather than their ordering. In well-developed hierarchies, such superpositions may be formed with some weights on the component-universe elements. Here, the objects from the two components coexist in the same moment, and we can employ all of them. Then every set A of the joint theory will be representable as a direct sum of the component sets separately produced by each universe: $A = A_1 + A_2$. This differs from the usual set union by that the two components never mix up, independently participating in any set-theoretical constructions. In particular, one cannot speak of the overall number of elements, but rather retain two numbers, one for each component. Certainly, some combinations (direct sums) of the sets over the incident universes U_1 and U_2 taken separately may be absent among the sets of the joint universe; the existence of the result is to be established in every particular case, accounting for the imposed constraints.

In respect to the elements of the sets, sequential synthesis could be considered as inner junction: every element of a set will be split in two components. Parallel synthesis acts in the opposite manner: the sets are taken as a whole and joint together without influencing their elements.

If $A = A_1 + A_2$ and $B = B_1 + B_2$, the union of the sets can be defined as component-wise: $A \cup B = A_1 \cup B_1 + A_2 \cup B_2$. Since there are no empty sets in an objective theory, every set above the joint universe will contain the both components. This is quite like a Cartesian product producing pairs of objects where there are the both elements, so that the positions in a cortege cannot be empty. That is why the product $A \cap B = A_1 \cap B_1 + A_2 \cap B_2$ exists only when there are common elements in the sets A_1 and B_1, A_2 and B_2, pairwise. Note that neither of the component intersections must necessarily exist in the hierarchies over U_1 and U_2 as separate universes; this does not prevent them from appearing in a composite set.

In general, the universe U can unite many components, with different junction types. In such hierarchies, it is especially important to complement formal constructs with an analysis of the existing constraints. The correctness of judgment, in such a theory, depends both on its inner logic and the nature of the object. Let S be, say, a universe of symptoms, C will denote the individual contraindications, and M will be the universe of therapy schemes. Then the sets over the composite universe $(S + C) \times M$ will represent the possible instances of medical treatment. Obviously, only a part of the possible combinations will make sense. For a constraint, one could take the overall dynamics of the disease, which is to be directed to the final recovery, rather than the other way. The specific dynamics of treatment will be representable in this scheme with a sequence of sets, to account for the possible changes in the patient's state.

For the sets over the composite universe, internal sets and classes (external sets) are define the usual way. Here, beside elementary classes, we will also get all kinds of projections, that is, the classes of set coinciding in several components. For instance, consider the parametrical family of sets $A = A_1 + *_2$, where * stands for an arbitrary set over the universe U_2. The class of all existing over U sums of that form (the projection onto A_1) determines a class over U_2, which could be called adjacent to class A; in the general case, such a class may be absent in the standalone hierarchy over U_2, and its intersection with the native classes of U_2 (if existing) will form the boundary of the family A in U_2. Similarly, the classes of Cartesian projections can be defined as the families of sets with the elements of the type $\langle x_1, *\rangle$ or $\langle *, x_2\rangle$. As for elementary classes, the whole hierarchy becomes reflected in itself, since thus obtained structures can be related to some of the already introduced constructs of the theory.

Of course, one is free to consider any other (logical) structures on the set level with the corresponding "classes of equivalence". Still, arbitrary formal constructs won't automatically become components of the theory; they yet need an objective interpretation. The correspondence between the classes and the objects from the universe can serve as a criterion of truth, limiting the applicability of formal operations.

To resume: every objective set theory 1) requires some (not necessarily formally definable) universe; 2) builds a set level over the universe and considers set classes; and 3) identifies the classes with the objects from the universe. The arrangement of the theory will thus reproduce the organization of the object.

August 2006

Cardinal Hierarchy

Everybody heard about the famous Kantor hierarchy of infinite cardinal numbers: the next level comes up in considering the set of all the subsets of an infinite set. One could enumerate these infinities with integers (finite ordinals): level-0 infinity corresponds to discrete sets, level 1 is reserved for continuum, all functions over continuum form level 2, *etc*. Of course one is free to discuss entities beyond this simple layered structure, up to considering uncountable infinite ordinals.

In these terms, the notorious continuum hypothesis states that there is no intermediate cardinality between levels 0 and 1. The statement is very strong, and it cannot be either proved or refuted within the traditional axiomatic set theory. One could observe that this essential discreteness is mostly due to the binary character of the common mathematical rationality, which determines the way we construct power sets. Still, nothing prevents us from extending the idea of a set just a little, retaining the usual notions as natural limit cases. There is a generalization that allows to constructively demonstrate the inappropriateness of the continuum hypothesis in the generalized theory.

Indeed, let us note that the elements of a set in the traditional set theory are joined in the set in an outer manner, as externally opposed to each other. Each element corresponds to the same counting unit, which is infinitesimal for infinite sets, though without losing its qualitative definiteness common for all elements; and this is why we can righteously compare then to each other and count them. Kantor hierarchy therefore provides a series of outer infinities.

Now, assume that the elements of a set are no longer simple counting units, and each element is internally structured. For instance, a single outer space point (an element) could incorporate some inner space characterized by an appropriate Kantor power. In general, the organization of the inner space may vary from one element to another. However, if we are to study some objective integrity, there are good reasons to believe that the mode of unfolding is the same for all elements, so that their inner spaces should be at least of the same cardinality; in many practically important cases (like mechanical motion) one could confidently impose the demand of the same topology.

In the simplest case, the inner space is discrete (similar to the usual spinor components); this is a level-0 subspace. However, each point may also be innerly represented with a continuous area, a kind of zone, which makes it a set of level 1. Such situations are quite common in real life. For example, the perception of a pure tone of some musical height (a logarithm of sound frequency) subjectively pictures it as a distribution of heights in the vicinity of a well-pronounced maximum. From this observation, one can derive that the possibly sets of discernible musical tones form zone structures (musical scales), where each element is far from being a single point, but rather a continuum of the admissible deviations from the center of the zone.[1] What is to be taken for the cardinality of such a hierarchical set? It is certainly discrete, while, on the other hand, it is a union of continuous intervals.

Let us define the cardinality (power) of a two-level set (with a uniform inner space structure) as a pair (K_1, K_2), where K_1 and K_2 indicate the cardinalities of the higher (outer) and lower (inner) levels respectively. In particular, inner space may be absent, which means that it effectively consists of a single element, and its level of cardinality is zero. With all that, a purely discrete set is characterized with the cardinality of (0, 0); a usual continuum has the hierarchical power of (1, 0); a discrete structure with an inner continuum should then be assigned the cardinality of (0, 1). These cardinalities can be naturally ordered in the lexicographic manner, first comparing the upper levels, then the lower levels (if needed). Obviously, $(0, 0) < (1, 0)$. However, as naturally, $(0, 0) < (0, 1) < (1, 0)$. That is, there is a hierarchical set with the generalized cardinality between discreteness and continuum.

No doubt, the process can be continued on and on, as the lower-level elements are considered as complex, in their turn. Sticking to the first two Kantor numbers, the cardinal number of an arbitrary hierarchical set can be represented with a sequence (b_1, b_2, b_3,…), where b_k are either 0 or 1. One can readily observe that this is equivalent to a binary notation for some real number in the interval (0, 1); consequently, there is an infinity of cardinalities intermediate between 0 and 1.

Of course we are free to consider much more complex inner hierarchies. For instance, the hierarchy of infinite ordinals can be reproduced in full. It is especially intriguing to consider all kinds of isomorphism between the spaces of different levels, which will bring us far beyond the trivial tree-like structures, with a number of circularities and loops. Calculating the cardinalities for such sets is a yet another interesting problem.

July 1994

The Quality of Negation

As philosophers declare, the primary purpose of the human (or any other) reason is binding the world together, connecting things that can in no way get connected otherwise. If so, the traditional mathematical habit of interrelating seemingly different entities should be respectfully appreciated as a part of the common productive work. Life gives many examples of how the tricks of one branch of industry perfectly match the needs of another; the mathematical language may come quite handy for sharing such fundamental schemes, and its practical importance is beyond any doubt. Still a sober attitude to the available resource is no less valuable. We are a part of the world, and any reflection of that whole in powerful abstractions, however accurate, is bound to remain partial; no formal construct can pretend to an unreserved universality. That is, our ability to tie one mathematical object to another, up to eliminating any distinction at all, can never guarantee that these entities would not accidentally get stuck in an entirely new environment requiring a clear awareness of their difference, and hence a different mathematics, albeit envisaging a quite decent retirement for earlier theories.

On the other hand, the multitude of peculiarities cannot appear but on the background of the primordial universality of the world, as there are no other worlds, and (therefore) no way out can ever open. In human practice (including mathematics), this results in the hidden presence of the sprouts of

[1] L. V. Avdeev and P. B. Ivanov, “A Mathematical Model of Scale Perception”, *Journal of Moscow Physical Society*, **3**, 331–353 (1993).

the future in the present and the past; every incidental guess reveals something that has long since been existing in the culture without attracting too much attention, silently waiting for its hour to come. Hence the typical technology of a scientific discovery: take a most plain and banal experience that would not even deserve mentioning in a good company, and pin up a couple of miserable subtleties that might eventually become great and meaningful. Those who are too lazy to format it as an academic hit may prefer the roads of the commonly known, with just a few philosophical deviations.

Well, here too, let us take a perfect commonplace and wonder if there is some underestimated creative potential in there.

Ask a not-yet-born baby ripening in a womb: *two minus three, how much is it?* You'll get a prompt answer: *minus one, of course!* Our lazy philosopher comes to press on with a real stumper: *and what's that, minus one?* The under-baby, scratching its head with a navel-string, thoughtfully mumbles: *well, it's a kind of number… a negative … exactly like one, but with the minus sign*. This is enough to conclude that we are talking to a future mathematician: the lay people are ordinarily devoid of such clarity of the mind. Most would call this subtraction exercise improperly posed: it is not allowed to subtract a greater number from the lesser. Some would concede to the school answer (of minus one), with an explicit reservation that such numbers do not really exist, referring to sheer convention oriented to meeting a substantial enough positive individual, to clutch on him and diminish by an appropriate amount. In natural languages, being negative is invariably associated with something bad and wrong, some crazy dysfunction that should not normally happen. And, of course, there is an ancient profession whose representatives are firmly convinced that there are no negative numbers at all, and all we may deal with is positive records in the books, entries on a number of accounts split into the pairs like "*we have — we need*", or "*sums due — payments to receive*"; active and passive records are distinguished accordingly: fortune estimate *vs*. accumulated debt.

It is bluntly stupid to ask about the truth of all these viewpoints; each is certainly right, within their specific experience, and each requires as specific mathematics. In the pre-scientific times, it was commonly accepted that any particular thing can be characterized with a bunch of distinctive qualities, so that quantitative distinctions could be discussed within each quality. That is, first decide on whether it is in there, or it is not, and then inquiry for how much, where there is some. The absence we denote as null, while the present quantity can be numerically expressed as soon as some units of measurement have been fixed; the number **1** refers to the chosen scale for the specific dimension. An ancient geometer knew for sure that a line segment possessed some length, which did not depend on the mode of construction and the spatial orientation. Similarly, figures have area, and bodies have volume; when there is nothing measurable, the notions of length, area and volume are no longer applicable, and their usage amounts to a logical fallacy.

With all that, people do not live by the present moment; they think about the past and the future. The past feeds memories; the future makes plans. One is free to judge in an emotionally human manner, regretting the absence of the former glory or impalpability of things to come. This sorrow is referred to in mathematics every time we put down the minus sign.

An important note: compared to mere expression of the absence of something, the idea of the negative is a drastic step forward. No mere nothingness, but rather a kind of presence, albeit in an ideal way, inside the subject (or in any physical bodies, as an imprint or possibility). Obviously, such an ideal existence is somewhat different from real presence, and one should not confuse the two, lumping positive and negative experiences together. They are measured in different units. Nevertheless, they also have something in common: the very air of definiteness, qualitative homogeneity, and hence measurability. Which means that the uniform treatment of the positive and the negative is justified by their similarity in a certain respect, that is, in the context of a specific human activity, as long as we do not run off at the mind and disregard any distinctions at all.

To moderate our math and avoid too formal an attitude to the problem, there are well-proven formal tricks. One of them amounts to the already mentioned separation of the debit/credit kind, which directly makes practically different things also theoretically different. The debit and the credit are opposites, and one negates the other, but this negation is no longer merely quantitative, as it combines different qualities. Somewhere else, for the higher management staff, there is no difference, with one big bulk directly subtracted from another. Still, this is a faraway view, while operative accounting is to deal with the old two columns in the book. Once again: this is not a relic of primeval primitivism, or a moss-frown tradition; this is how the world goes round. Thus, biologically, everybody has a father and a mother; still, some children are brought up by a single mother, so that one parent is missing and comes with the negative sign; however, this differs from the actually having a single parent (for instance as in instance of parthenogenesis). That is, 2–1 is *qualitatively* different from the plain unity. It is only for school mathematics that all's the same.

In this line, monitoring resources will put everything that has ever come on the positive side as a present-tense fact. However, a part of that income has probably been partially consumed and is only present in memory; this is a past-tense negativity. Further, our current needs form a future-tense minus, that can significantly overweigh the present abundance. Note that positive numbers basically refer to the objective state of things, while negative contributions relate subjective feelings (at least within a single level of hierarchy); this may justify the position of those who consider negative numbers as merely conventional.

To account for conventionalities, a mathematician can represent a "generalized" number s with a cortege of two components $\langle a, b\rangle$, where a and b are non-negative real numbers (as presumably well-definable). The components a and b give, respectively, the positive and negative part of the number s which will conventionally be referred to as an *additively split real number*, or simply a *split*, to save space an the reader's effort.

Can we find anything like that in the history of mathematics? At every corner! For instance, rational numbers are traditionally introduced as pair of integers, with the rules of combining the components adjusted to our vision of the ordinary arithmetic. Similarly, complex numbers are defined through their real and imaginary components, with an appropriate specification of arithmetic. If so, why shouldn't we follow the same well-beaten track to play with real negation?

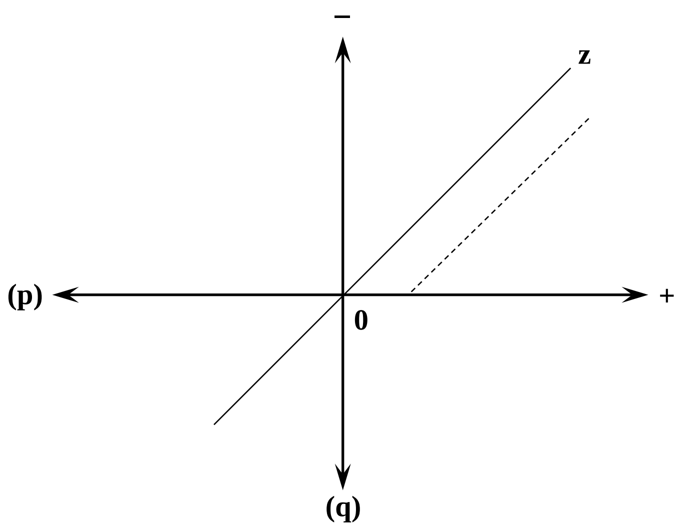
Treating the components of the cortege as the coordinates of a point in the plane, we effectively chose the positive and negative unity as the basis vectors of this space, denoting them as (+1) and (–1) respectively. Here, formally, (–1) is a monolith pictogram for some measurement unit, which may have nothing to do with the units used in the positive scale. Conversion of one unit to another will require an appropriate dimensional factor.

By analogy with complex numbers (and linear algebra), a quasi-algebraic notation could be employed: $s = a + (-1)\,b$, with the basis vector (+1) traditionally omitted for brevity, but always implied. A minimal list of useful features of real splits could be illustrated by the table below, where the complex-plane analogs are shown as well.

splits	***complex numbers***
$(-1)^2 = 1$	$\mathrm{i}^2 = (-1)$
$1/(-1) = (-1)$	$1/\mathrm{i} = (-1)\mathrm{i}$
$s = a + (-1)b$	$c = a + \mathrm{i}b$
$a = \mathrm{Pos}\ s$	$a = \mathrm{Im}\, c$
$b = \mathrm{Neg}\ s$	$b = \mathrm{Re}\, c$
$s = \mathrm{Pos}\ s + (-1)\,\mathrm{Neg}\ s$	$c = \mathrm{Re}\, c + \mathrm{i}\,\mathrm{Im}\, c$
$s_1 + s_2 = (a_1 + a_2) + \mathrm{m}(b_1 + b_2)$	$c_1 + c_2 = (a_1 + a_2) + \mathrm{i}\,(b_1 + b_2)$
$s_1 s_2 = (a_1 a_2 + b_1 b_2) + \mathrm{m}(a_1 b_2 + b_1 a_2)$	$c_1 c_2 = (a_1 a_2 + (-1) b_1 b_2) + \mathrm{i}\,(a_1 b_2 + b_1 a_2)$
$\mathrm{Pos}\ (-1)s = \mathrm{Neg}\ s$	$\mathrm{Re}\ \mathrm{i}c = (-1)\,\mathrm{Im}\, c$
$\mathrm{Neg}\ (-1)s = \mathrm{Pos}\ s$	$\mathrm{Im}\ \mathrm{i}c = \mathrm{Re}\, c$
$(-1)s = b + (-1)a$	$\mathrm{i}c = (-1)b + \mathrm{i}a$

Of course, from the formal aspect, these equations are interdependent; however, our purpose is to stress the core of the approach, rather than adhere to rigor. Anyway, a deductive arrangement is entirely dependent on the strategy of arithmetization; for one possibility, one could depart from the algebra of splits with no reference to their inner structure (corteges and components); the positive and negative parts of a split will then appear in theory as functionals (mapping the space of splits into the space of positive reals), or as projectors (establishing correspondence between different splits). Since we keep on the tradition of building complex numbers over the real field, the negative unity naturally penetrates the rules of complex arithmetic. Still, nothing prevents us from *defining* (–1) as i^2; in this case, the behavior of splits is to be derived from complex numbers.

Since the non-negative real axis is already well-ordered, all splits can be divided into two classes: those with $a > b$ (Pos s > Neg s) are called positive; those with $a < b$ (Pos s < Neg s) are called negative. This definition does not imply any direct combination of the components; geometrically, we mean the subdivision of the whole space of splits (the first quadrant of the Euclidean plane, the area (**+|0|–**) in the figure below) into separate subspaces, lying above or below the main diagonal (**0|z**). Any other subdivisions for all kinds of practical purposes are as contrivable.



In the algebraic notation the negative unity (–1) may acquire the meaning of an operator (commonly known as *negation*) producing splits from other splits by the simple rule: the positive and negative parts of the split get interchanged. Graphically, this means reflection with respect to the main diagonal; in particular, the points of the axis (**0|+**) thus get mapped into the points of the negative axis (**0|–**). In the same manner, the positive unity can be treated as the operator of identity leaving each point as it is. In the component system, the positive unity is represented by the cortege ⟨1, 0⟩, and the negative unity by the cortege ⟨0, 1⟩. However, the (positive or negative) unity as an operator differs from the corresponding unit vector; negation as an action and as an action's result means different entities.

The transition from the component representation of splits to polar coordinates is formally the same as with complex numbers:

$$s = r\left(\cos\varphi + (-1)\sin\varphi\right)$$

This layout may be useful to discuss certain intricate structural aspects; however, its justification can only come from a particular choice of relevant symmetries and metric. As we know, the traditional symmetries of the complex plane essentially differ from those of real splits. This is obvious from the very fact that splits are define in a single quadrant of the plane, so that extending theory to the whole plane would require introduction of yet another pair of unit vectors (for instance, p and q), as shown in the figure; thus generalized splits would then be formally written as a+(–1)b+px+qy, which resembles the quaternion structure. Imposing additional symmetries will "glue" some points together, folding the space in a specific manner. For common real splits the fundamental symmetry is

$$\langle a,b\rangle + \langle z,z\rangle = \langle a+z, b+z\rangle = \langle a,b\rangle$$
$$(a+z)+(-1)(b+z) = a+(-1)b$$
$$s+(-1)s = 0$$

where *z* is a positive real number. Graphically, this identifies all the points of the main diagonal, and any straight line parallel to the main diagonal forms a class of equivalence, effectively collapsing into a single point. The complex-plane analog

$$c + \mathrm{i}^2 c = 0$$

is quadratic in the imaginary unity, which naturally leads to the usual quadratic metric. By analogy with the orthogonality of vectors in the Euclidean plane related to the zero value of their scalar product, the equation $1+(-1)=0$ could be interpreted as a kind of additive orthogonality of the positive and negative dimensions.

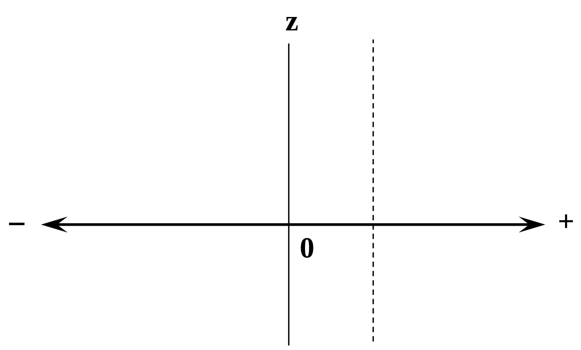
Any symmetries of a general system (or the imposed constraints) will often diminish the number of its degrees of freedom, thus changing the effective dimensionality. For splits, we formally reduce the two-dimensional (or even four-dimensional, as in the "quaternion" extension) picture to a single dimension; this is an essentially nonlinear operation akin to projection. There are other types of projections, like the extraction of the positive or negative (real or imaginary) part, calculating the (quadratic or linear) norm, determining the phase *etc*. In ordinary life, we always observe any system in one of the possible projections, so that the existence of the others is to be somehow deduced. This is exactly like the choice of the gauge in relativistic physics. The unavoidability of projections is related to the complexity of inner motion; however, the symmetry of this unobservable behavior is determined by what we really can do with the system, and what is to be considered as the outcome.

The effective one-dimensional structure of the globally symmetrized splits should not deceive anybody: equivalence does not mean absolute identity. The bunch of straight lines parallel to the main diagonal is not the same as a collection of points in a single line, and there is still the qualitative distinction of the shifts up and down from the main diagonal (the regions with $a < b$ or $a > b$).

Under the global symmetry (equivalence), for any split *s*, there is a unique equivalent "reduced" split $R(s)$, such as either Neg $R(s) = 0$ or Pos $R(s) = 0$ (the positive and negative reductions). A linear order on the entire space of splits could be introduced, agreeing that any positive split is greater than any negative, and the positive and negative subspaces are independently ordered by the increase of Pos $R(s)$ and the decrease of Neg $R(s)$, respectively:

$$\mathrm{Neg}\, R(s) = 0 \,\&\, \mathrm{Pos}\, R(s') = 0 \Rightarrow s > s'$$
$$\mathrm{Pos}\, R(s) > \mathrm{Pos}\, R(s') \Rightarrow s > s'$$
$$\mathrm{Neg}\, R(s) > \mathrm{Neg}\, R(s') \Rightarrow s < s'$$

Consequently, the positive real axis can effectively be continued into the negative domain, which may look like transition from the two-dimensional picture to a single dimension. Graphically, the first quadrant is thus effectively expanded to the upper half-plane, with reduction being visualized as projection onto the horizontal line $(-|+)$:



Still, these formally defined positive and negative branches yet remain separate and independent. Yes, each can be mapped onto the other, retaining the algebraic structure within the subspace; however, it is not quite evident how operations between the elements of different subspaces are to be introduced, to make the entire combined space uniform enough. One can easily observe that

$$R(s_1 + s_2) = R(R(s_1) + R(s_2))$$
$$R(s_1 s_2) = R(R(s_1)R(s_2)) = R(s_1)R(s_2)$$

While multiplication of reductions keeps within the same straight line, adding together reductions of the opposite sign will certainly violate this simple structure, and one has to recourse to yet another reduction. That is, constructing the completely coupled field of real numbers implies rather strong assumptions that do not necessarily conform with the nature of applications.

A too straightforward interpretation of the quasi-algebraic notation for splits (or complex numbers) may lead to confusion, as the qualitatively different objects get treated on the same grounds. It is important to remember that the character combination $s = a+(-1)b$ is only a different graphic expression for $s = \langle a, b\rangle$; the characters a and b refer to real numbers, (–1) points at the position in the cortege, while the plus sign means nothing but considering the two parts together. To produce an algebraic expression proper, one would write something like

$$\langle a,0\rangle + \langle 0,1\rangle \cdot \langle b,0\rangle = \langle a,b\rangle$$

Similarly, in the right-hand side of the algebraically interpreted constraint $s+(-1)s = 0$, the character 0 does not stay for real zero, but rather for the cortege $\langle 0, 0\rangle$, so that

$$\langle k,k\rangle = \langle 0,0\rangle \neq 0$$

The placement of the positive and negative axes on the same straight line does not change anything in that respect, while rather hiding and disguising the (additive) orthogonality. An analogous symmetry exists for rational numbers represented with the corteges of two integers:

$$\langle k,k\rangle = \langle 1,1\rangle$$

since we can cancel the common multiplier in the numerator and the denominator of the ratio. Something like that can also be written for complex numbers brought to polar coordinates, $\langle x, y\rangle \to \langle r, \varphi\rangle$. In the absence of ramification, there are two conventional symmetries

$$\langle r,\varphi\rangle = \langle r,\varphi + 2\pi k\rangle$$
$$\langle 0,\varepsilon\rangle = \langle 0,0\rangle \neq 0$$

for any integer k and real angle ε. However, these symmetries may be violated in certain cases, so that a significant phase shift can bring us onto a different branch, and approaching zero from different directions can result in different limit values.

To summarize, in any theory, null (as an origin of a scale) and none (as negligible quantity) refer to the absence of a specific quality rather than to the absence of anything at all. In other words, there is no zero in general, there are many qualitatively different zeros. That is why, under certain assumptions, we can extend operations with the objects of the same kind and include the corresponding zero object in the elementary base. With all that, a rigorous mathematics would not treat zero as a number (or any other singular object), but rather as a mode of object production, something generic: a pattern, a scheme, a template, a type, the class of objects as the expression of the very possibility of their differentiation. Thus, the symmetry $\langle k, k\rangle = \langle 0, 0\rangle$, which can be interpreted as the equivalence of the point of the main diagonal, can also be turned inside out to describe the emergence of virtual pairs from null resembling the physical vacuum (which is far from being a sheer emptiness!). Pulling out a particle from the vacuum, we also create a hole of the same kind and size. Just like in physics, the zero level may often be movable; however, this symmetry does not influence the shape of the system in respect to the others.

In the same way, infinity is not a number (albeit fancied as infinite cardinal or ordinal), but rather the very activity of distinguishing the opposites and translating one into the other. Zero is related to infinity like the possibility of a product is related to actual production. A computer program is not the same as its execution; however, programming languages always account for hardware architecture, as well as hardware gets gradually adapted to language idioms.

Note that the special role of the null has to do with the logic of theory. Thus, one can assert the truth of a statement (+1), or its falsity (–1); either of these values is logically definite, assuming the possibility of verification (or falsification). On the contrary, the zero value will rather mean that the problem has been ill-posed, as the categories of truth and falsity are no longer applicable to the case (though the same statement, in this context, may allow positive or negative evaluation in some other respect, for instance, being correct or incorrect).

In the theory of splits, zero components stay for the absence of any operations on the positive or negative side, “debit” or “credit”. Considering some inevitable overhead, the cortege $\langle 0, 0\rangle$ is not fully replaceable with a virtual exchange $\langle k, k\rangle$ (writing off the same amount as placed to the account). Such operations may be nontrivial when the units of the positive and negative components differ (say, like in currency exchange). However, in classical bookkeeping, movements like that are often implemented as a pair of reductions $\langle a, 0\rangle$ и $\langle 0, b\rangle$ put on different accounts, instead of the split $\langle a, b\rangle$. The distinction of the two technologies resembles the difference between correlated quantum transitions and cascades; the interference of virtual processes may result in clearly observable (“macroscopic”) effects, but any financial speculation is organized in exactly the same manner! On a higher level of hierarchy, in consolidated reporting, the formal addition of movements recorded on different accounts is admissible:

$$\langle a, 0\rangle + \langle 0, b\rangle = \langle a, b\rangle$$

Yet another level involves a positive or negative reduction (or the zero balance).

The profound mathematical (and philosophical) sense of splits is due to the fact that every object can be produced in many ways, which may be considered as equivalent in some respect, while revealing some important differences in another context. Here, additive real-number splits have been discussed for illustration: every number can be virtually represented by a difference of two other numbers. Multiplicative splits are structured in the same way, as any real number can be treated as a fraction; such splits become additive in a logarithmic scale. Similarly numbers can be split into sums or products (possible infinite). Just take the example of the decomposition of any integer into a product of primes, forming the core and purpose of the classical number theory. More examples: rotation in the opposite directions, the difference of outer and spinor dimensions, layered manifolds… Finally, the very complementarity of the object and the subject in the context of a definite activity is of the same split nature. Since every activity is to reproduce its product on a regular basis, splits can be understood as loops, cyclic paths. Obviously, many-component and many-level splits are as feasible. In the general case, any mathematical object implies a hierarchy of all the possible splits, the modes of definition. The expansion of this hierarchy into an admissible hierarchical structure is known as a mathematical theory. The direction of this development is always suggested by the practical considerations, coming from the current human needs. Mathematics (as any other science) serves people to assimilate some formal techniques, the typical schemes of activity, to free our reason from the routine operations for striving to the yet unexperienced.

Dec 1983

Oriented Curves

In the early years, everybody must once have played with the Moebius strip. It’s a really amazing thing, while the mathematical theory behind it is not of an entirely formal kind. The Klein bottle is much simpler, as it is a regular 2-dimensional surface in a 4-dimentional space, quite smooth and posing no conceptual problems. A surface with an edge is already a challenge, since there is no way to define the edge of a manifold from within, in its own terms; we need to get out, to relate the inside to the some outside, so that much will depend on choice of the embracing space and the mode of embedding. What holds for one case may not be applicable to another. That is why rigorous reasoning on singular spaces do not seem generally convincing, as there is always a feeling of an *ad hoc* theory stretched to the experiences to produce. Which, however, does not prevent science from being quite entertaining and instructive.

Can we get rid of singularities and switch to a simpler construct? Traditionally, topologists employ gluing; but, since the Moebius strip has a single edge, it is not evident how it could be glued. With yet another standard trick, regular contraction, one can reduce the edge down to a (puncture) point; the result is still singular, though it might be considered as simpler, in a sense.

Still, there is a different option. Note that the width of the Moebius strip does affect its topology. So, let us make the strip infinitely narrow thus making it into a closed curve. In this way, we get a (one-dimensional) manifold without edges; the outer peculiarities of the strip will become the inner structure of the curve. The Moebius strip can be produced gluing the ends of a regular ribbon with a half-turn on one of the ends. Now, we have to clarify what such a warp mean for a (spatial) curve to close.

Just visually, each point of any curve can be assigned with a (three-dimensional) orientation vector orthogonal to the direction along the curve; when we go from one point to another, the orientation vector will, in general, turn in its space. Gluing the ends of the curve with the same orientation, we get a regular space curve, which could be projected onto a plane so that the inner and outer regions of the projection would be clearly distinct, and the notions of inner and outer normal could be introduced. Gluing the ends with opposite orientation will produce the analog of the Moebius strip; in the projection on the plane, the outer normal will abruptly become inner after a full turn, and this behavior does not depend on the starting point. The apparent singularity is in no way related to the smoothness of the manifold: this is an artefact of the essentially nonlinear operation of projection and vector normalization.

Formally, there is an orientation system in each point of a spatial curve, one of the dimensions corresponding to the direction along the curve, another follows the transversal displacement (for a strip, this means the local motion from one edge to another; their vector product gives the position of the (three-dimensional) normal. Such an orientation system will generate the inner space of each point, to be distinguished form the outer, attached spaces (for instance with the axes along the local velocity and the radial acceleration); in general, the outer characteristics of the curve (like curvature and torsion) are not related to inner properties (the position and structure of the orientation frame).

Since the inner space does not depend on the outer, a displacement along the curve may, in addition to transversal orientation change, may also switch the very sense of the direction along the curve. In a three-dimensional embedding, this may produce an impression of singularity, cusp, retrograde motion. Still, embedding the same curve in four dimensions can preserve the uniform smoothness, so that the apparent irregularities could be explained by the choice of projection. There are many common-life examples. For instance, the mathematically smooth motion of a point pendulum shows up as a retrograde motion at the high ends of the trajectory; we also know that the motion of a distant airplane, when projected in the observation field, may produce weird curves that some people take for the maneuvers of a UFO. This is quite common in physics, when some nonlinear effects and singularities can be interpreted as the presence of hidden dimensions, up to the conjectures that the exustence of the light barrier (the impossibility of higher-than-light speeds) or the Schwarzschild singularity might hint to the higher dimensions of the physical space, where any movements are possible while the three-dimensional projection effectively crops the observable range.

The difference of the outer and inner spaces might be compared to the distinction of physical fields from geometry. The orientation of the inner space axes does not depend on the transforms of the outer coordinates (including mirror reflection)$ this is a formal expression of the independence of a physical system from the observer.

Displacement along the curve and the rotation of the normal in the plane orthogonal to the direction of motion can be characterized by two angle parameters: phase and orientation proper. If orientation gets modified by $2\pi k$ with the phase incremented by 2π, the curve corresponds to a regular band; with orientation shifted by $2\pi(k + 1/2)$, this is an analog of the Moebius strip (with the half-integer number of twists). In the latter case, the curve gets effectively split into two layers (or sheets), consecutively spanned during each 4π turn. One could normalize this extended loop: $4\pi \to 2\pi$; this will make the Moebius curve into a regular curve. This obviously corresponds to the operation of slicing the Moebius strip in the longitudinal direction (which is known to produce a twisted regular strip). In general, the dynamics of orientation rotation may differ for the two halves of the extended loop; this difference does not matter in the absence of constraints, as it can be eliminated by the redefining of the "time" variable.

In the general case, a phase shift by 2π will result in the orientation change by $2\pi q$, where q is an arbitrary real number. For rational q, the curve will split into a finite number of layers; irrational factors will produce a kind of toroid. When the dynamics of orientation change depends on the phase, the spectrum (the density of curve turns around the layer surface) may exhibit quite nontrivial variations.

The rotation of orientation along a closed curve could be illustrated by a physical model with an orthogonal to the motion direction dipole in every point of the curve, so that the polarization of the next point depends on the previous. A finite number of layers then would correspond to a standing wave, while the non-periodic orientation changes describe a wave running along the curve. Beside some exotic interpretations (like treating the magnetic monopole as a Moebius strip), there are practically useful applications as well.

Oriented curves are hierarchical structures where each point unfolds into an inner space. The outer motion along the curve is naturally associated with a number of outer layers, with the curve (as a configuration space) becoming a stratified manifold. When it comes to the projection of such a construct onto a plane, it is not enough to merely establish correspondence between the point of the curve and the point of the plane: the inner and outer stratification too are to be reflected in the projection. This is possible in certain cases; for instance, the plane could be split in a number of sectors, each hosting a separate branch of the curve. More often, some features of the whole will be lost in projection. Similarly, higher-dimensional manifolds could be projected onto simpler spaces with a loss of important detail.

The levels of hierarchy essentially depend on each other. Consider a trivial mathematical illustration: let an oriented curve be represented by a narrow plane ellipse whose axes rotate as we move from one of its point to another; if, after a full span, the ellipse will take the original position, we deal with a regular curve.

An oriented curve is only apparently one-dimensional. In principle, all kinds of hierarchical structures could be unfolded in each point, while the transition from one point to another would mean folding one hierarchical structure and unfolding a different one (hierarchical conversion). In particular, the dimensionality of the inner space may change along the curve, with the same geometry of the embedding.

Jan 1985

Virtual Mathematics

In quantum physics, we are used to refer to the inner states of some system (the points of the configuration space) which are never observable in any direct manner, so that the only way to guess about their presence is to interpret the outer behavior of the system the right way: the observable sequence is organized *as if* there were such inner states interacting in a definite way. Still, nothing prevents us from computing the same observable effects using an alternative technique that would not require any idea of the formerly introduced virtual states. This is a regular situation: in everyday life, we often produce the same from different components, using all kinds of tools; so, why not build a quantum system on a different basis?

There are mathematical analogs of this virtuality, and most often they are related to the formal manipulations outside the region of their applicability. Fundamental mathematicians prefer to ignore any objective foundations, pretending that mathematical constructs exist on themselves regardless of any possible applications. This is nothing but fantasy, or self-delusion. Every mathematical theory presumes a definite universe to support the abstract forms of a certain class. A rigorous derivation cannot mean anything but the elements of its object area, so that reasoning about any outer objects would be a logical fallacy. Nevertheless, in a purely formal manner, we can introduce objects that are impossible (virtual) in that particular theory, as a good occasion for passing to a different (extended) theory incorporating such abstractions along with the other objective features.

For example, adding up two natural numbers, we obtain a natural number that is greater than any of the original items. That may be interpreted as the (virtual) presence of these items in the sum as its inner components. Fancy a theory knowing nothing beyond even numbers (which will therefore constitute its universe, the object area). In this theory, a representation of an even number with a sum of odd numbers is closely resembling the idea of a quantum amplitude as a superposition of some virtual basis states. Going yet farther, we can admit the existence of such entirely exotic entities as negative numbers that will diminish any original item in addition. However, as soon as we learn to produce things representing these fantastic creatures, one can raise them to the class of observables and consider all kinds of integers together.

Similarly, a product of positive integers in a positive integer; taken together, all such numbers form the object base of one of the most important mathematical theories, where the operation of expanding an integer number in a product of integers plays a most fundamental part. Every natural number is then representable with a cloud of virtual products. There are “prime” numbers that reduce that virtual cloud to minimum; they provide a kind of a basis for the whole universe. For each number, its inner structure can be described by a sequence of the “level sets” L_n containing all the products with n components (from which we exclude the unity factor 1 since it, in fact, only specifies the scale, the units of measurement). The first level will obviously contain only the original number, while the lower level are absent for primes. Denoting the number of elements (the measure) for a set X as $\mu(X)$, we can introduce the *index* of a natural number N as

$$\lambda(N) = \sum \frac{\mu(L_n)}{n!}$$

For any prime number, this value obviously computes to 1; all the other numbers will produce values greater than unity. For instance, $\lambda(15) = 1.5$, and $\lambda(12) = 3.5$. The order of components in the product is important, but here, we do not distinguish equal factors (admitting that a different definition might be useful elsewhere). The factorial weighting has been introduced from real life considerations: the variants of less lengths are more practical, producing the same thing with minimum effort. The index could characterize the hierarchical complexity of a natural number; on the other hand, it is related to the practical “productivity”, since a number with a greater index can be obtained in many alternative ways. Given the expansion of a natural number into the product of primes, one cam easily calculate its index; however, the expression is no too trivial, which brings up observable effects similar to quantum interference, mixing the different inner representations (reaction channels).

Just like in the above additive example, virtual negative numbers could be introduced as well, possibly treated as positive numbers multiplied by the “negative unit” factor (–1) similar to common dimensional component (+1) in all the positive integers. The “physically allowed” virtual trajectories will therefore be restricted to the product expansions with the even number of negative contributions; alternatively, one could say that negative dimensions (factors of –1) in the positive nature can only be born in pairs (like the poles of a magnet). A considerable mathematics can be developed about such multiplicative universe, with nontrivial generalizations of the index theory.

The transition to the theory of all the integers will virtually retain this complexity; still, it may no longer be that significant, considering the highly symmetric nature of the general algebraic structure arising from the assumption that negative numbers can be “observable” on themselves (that is, they can be represented with some palpable things.

Similarly, in the theory of real numbers, no computation can be considered as "physical" unless it is going to produced a read answer. The introduction of the imaginary unit as $i = \sqrt{-1}$ is an obvious violation of the domain of the square root function; however, complex-valued expressions are quite admissible as virtual paths, provided we get something real in the end. Here, the analogy with quantum virtuality is much more pronounced: the same real number can be approached by many paths in the complex plane, so that every real number becomes a hierarchy of complex loops, or cycles; a similar introduction of the hierarchical complexity index allows to develop an extensive mathematical theory. In the presence of constraints, the topology of the object area becomes more complicated, and the existence of at least one cycle (albeit of the zero length) can no longer be guaranteed. The virtual character of the imaginary unit within the real-number theory is also a kind of constraint limiting the collection of the admissible paths. On the other hand, such a theory could consider some alternative topologies of the complex plane not employing the traditional operations of complex addition and multiplication. Still, if we are going to build a uniform theory of complex numbers, we will need to fix the appropriate rules thus choosing one of the possible (though irreducible to each other) structures.

Yet another example: negative and complex sets in a theory, where each set is associated with the ways of its virtual production, construction from other sets.

In the exactly the same manner, non-traditional logical theories could be developed, with the classical truth valuation obtained on the basis of non-classical reasoning.

In general, any branch of mathematics admits both classical theories, as working exclusively with the objects definable within the theory, and various "quantum" extensions accounting for the different modes of virtualization.

Dec 1982

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