The Formal Theory of Scattering

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The theory of scattering is developed from first principles with strict attention to the question of the preparation of the state vector of the system appropriate to a description of scattering. The connection between the present formulation and the more conventional interaction representation and S matrix presentations is traced. The wave matrix of Møller is introduced and the existence of bound states is discussed in connection with it. A number of applications to rather involved processes are discussed. Finally, the problem of self-energies in field-theoretic scattering calculations is treated.

I. INTRODUCTION

THE theory of scattering has become very familiar to physicists during recent years and the subject is discussed in numerous textbooks. Nevertheless, there does not appear to be a unified treatment of scattering theory which proceeds from fundamental quantum-mechanical principles and which is of sufficient generality to cover most cases of interest. The present work represents an attempt to fill this deficit. There are very few new results, but the point of view of the presentation is somewhat new.

The material to be presented is the outgrowth of a series of lectures on special topics in quantum mechanics given by the authors during the Spring Quarter (1952) at the University of Chicago.

The closest parallel to our development is to be found in the work of Lippmann and Schwinger.[1]

II. DERIVATION OF THE TRANSITION PROBABILITY

In a quantum-mechanical description of scattering, a system of two (or more) colliding parts is governed by a Hamiltonian H that includes interaction between them. We imagine, at least in simple cases, that H is split into two parts, which we shall call K and V, such that if K were the entire Hamiltonian the colliding parts would have the same internal structure but would suffer no scattering. The question we ask then is the following: What is the rate of transition from one such noninteracting state to another? From the transition rate, cross sections may be computed in the well-known way.

There are many problems, particularly in the nonrelativistic domain, in which the separation of H into K and V is trivial. K may be the kinetic energy or the kinetic energy plus the potential energy between a pair of particles. In any case, so far as the continuous spectrum is concerned, the eigenvalues of K and H are the same and there is no question of self-energies or of renormalization. For the present we shall consider such simple systems, postponing to Sec. V the discussion of the more complicated situations that arise in the case of quantized fields where the concept of noninteracting systems is rather obscure.

In the case to be considered first, the major problem is the characterization of the state vector of the system and its development from some sort of initial configuration. The fact that the interaction, V, is always present, but nevertheless the physical process is to be described in terms of the noninteracting state vectors, necessitates a very careful discussion of this point.

The Schrödinger equation (with $\hbar$ taken as unity) for the system with interaction is

$$i\partial\Psi(t)/\partial t=(K+V)\Psi(t). \tag{2.1}$$

Let us denote by $\Phi_i(t)=\phi_i e^{-iE_i t}$ the stationary state solutions (normalized to unity) of the Schrödinger equation in the absence of interaction:

$$i\partial\Phi(t)/\partial t=K\Phi(t). \tag{2.2}$$

We shall discuss the calculation of the differential cross section for scattering from state Φ_j to state Φ_i caused by the interaction V. The "initial state" Φ_j serves to characterize the actual state Ψ_j of the real system. We may, knowing Ψ_j, find the rate of increase, during the time of the scattering, of the probability that the real system is one of the "final states" Φ_i.

Suppose that we examine the transition rates at time $t=0$. It is necessary to represent mathematically the way in which the state Ψ_j has been prepared during times $t<0$, for example by directing an approximately collimated, approximately monoergic beam of particles at a scattering center. One might try a model in which, at some time T in the distant past, the system was in the "free" state Φ_j, so that $\Psi_j(t)=e^{-iH(t-T)}\Phi_j(T)$. However, undesirable transients are introduced into the temporal dependence of Ψ_j by the somewhat unphysical assumption that the train of incident waves is released all at once at time T. Rather, one must represent the incident train as fed in over a period of time in the past, using a Ψ_j that is a sum or average over T of the ones suggested above. For instance one could take $\Psi_j(t)$ as

$$\tau^{-1}\int_{-\tau}^{0}dT e^{-iH(t-T)}\Phi_j(T) \quad \text{or} \quad \tau^{-1}\int_{-2\tau}^{-\tau}dT e^{-iH(t-T)}\Phi_j(T),$$

with τ allowed to approach $+\infty$ at the end of the calculation. We shall adopt the form that is most

[1] B. Lippmann and J. Schwinger, Phys. Rev. 79, 469 (1950).

convenient mathematically; namely,

$$\Psi_j^{(\epsilon)}(t)=\epsilon\int_{-\infty}^{0}dT e^{\epsilon T}e^{-iH(t-T)}\Phi_j(T). \tag{2.3}$$

Again, ϵ^{-1} will be allowed to approach $+\infty$ in the evaluation of cross sections; but care must be exercised in passing to the limit since there is another limiting process to be performed. Our ϕ's are normalized to unity in a large region of linear dimension L and volume L^3.[2] We must remember that τ or its equivalent ϵ^{-1}, corresponding as it does to the length of the incident wave train divided by the group velocity v, may not exceed Lv^{-1}. When ϵ^{-1} and L both tend to infinity, quantities proportional to $\epsilon^{-1}L^{-3}$ will tend to zero.

It may be objected that we have unnecessarily restricted our choice of an average by taking the phases of contributions from various times T to be equal; that is to say, we have not considered such an expression as

$$\epsilon\int_{-\infty}^{0}dT e^{\epsilon T}e^{-iH(t-T)}e^{i\alpha(T)}\Phi_j(T),$$

with $\alpha(T)$ neither equal to nor finally tending to a constant. But it is clear that a variation in phase of the parts of the incident wave train over the length of time ϵ^{-1} corresponds physically to incoherence of those parts and would be incompatible with the condition that the energies in the beam be within ϵ of E_j.

Let us now proceed to the formal computation of transition rates. The probability that the system is in state Φ_i at time t is

$$w_{ij}(t)=|f_{ij}(t)|^2N_j^{-1}, \tag{2.4}$$

where

$$f_{ij}(t)=\langle\Phi_i(t)|\Psi_j(t)\rangle \tag{2.5}$$

and

$$N_j=\langle\Psi_j(t)|\Psi_j(t)\rangle. \tag{2.6}$$

(We have suppressed the index ϵ attached to the state vector.) The normalization N_j is independent of time because the Hamiltonian is Hermitian.

Equation (2.3) tells us that

$$\Psi_j(t)=e^{-iHt}\epsilon\int_{-\infty}^{0}dT e^{\epsilon T}e^{i(H-E_j)T}\phi_j, \tag{2.7}$$

or

$$\Psi\ (t)=e^{-iHt}\frac{\epsilon}{\epsilon+i(H-E_j)}\phi_j. \tag{2.8}$$

Since ϕ_j is an eigenfunction of K it satisfies

$$(H-E_j)\phi_j=V\phi_j, \tag{2.9}$$

and thus for the state vector at $t=0$ we have the expression,

$$\Psi_j(0)=\phi_j+\frac{1}{E_j-H+i\epsilon}V\phi_j. \tag{2.10}$$

Instead of the explicit formula (2.10) it will be convenient to use the implicit relation,

$$\Psi_j(0)=\phi_j+\frac{1}{E_j-K+i\epsilon}V\Psi_j(0), \tag{2.11}$$

obtained from (2.10) by algebraic manipulation. If V is to be treated as a small perturbation, one uses the power series expansion of (2.10) or (2.11),

$$\Psi_j(0)=\phi_j+\frac{1}{E_j-K+i\epsilon}V\phi_j+\frac{1}{E_j-K+i\epsilon}V\frac{1}{E_j-K+i\epsilon}V\phi_j+\cdots, \tag{2.12}$$

which also serves to show the connection between (2.10) and (2.11).

Using (2.11), we see that

$$f_{ij}(0)=\delta_{ij}+\frac{1}{E_j-E_i+i\epsilon}R_{ij}(\epsilon), \tag{2.13}$$

where

$$R_{ij}(\epsilon)=\langle\phi_i|V|\Psi_j(0)\rangle. \tag{2.14}$$

The form (2.13) is useful because it exhibits the nature of the singularity in f_{ij} when $E_i=E_j$ and ϵ tends to 0. In order to see that $R_{ij}(\epsilon)$ behaves smoothly with respect to energy when we pass to the limit, it is sufficient to substitute for $\Psi_j(0)$ in (2.14) the series solution (2.12). The Green's function,

$$G^{(+)}(E_j)=\lim_{\epsilon\to 0^+}\frac{1}{E_j-K+i\epsilon}, \tag{2.15}$$

appears only between V's and never acts directly on an eigenfunction of K. In field-theoretic applications, one must discuss the energy dependence of $R_{ij}(\epsilon)$ more carefully (see Sec. V). However, we must not apply the limit on ϵ to $R_{ij}(\epsilon)$ directly since, on account of the normalization of the ϕ's, $R_{ij}(\epsilon)$ is proportional to L^{-3} and the two limits must be taken together. (For the sake of simplicity, we shall treat the quantization volume as it would appear in a reaction in which two particles collide and two particles emerge. If there are more than two particles in the final state, appropriate factors of volume must be inserted. None of our conclusions are modified by this complication.)

We will take it for granted from now on that

$$\lim_{\substack{\epsilon\to 0^+\\ L\to\infty}}R_{ij}(\epsilon)L^3=\mathfrak{R}_{ij}$$

is free of singularities at $E_i=E_j$.

[2] Because of the fact that we are considering the system to be enclosed in a large box, the energy eigenvalues of K do not precisely coincide with those of H. The energy shift, which is of the order of magnitude ϵ, would, strictly speaking, require a slight modification of our formalism, but we shall disregard it for the time being. (See Sec. V.) None of our conclusions will be modified.

In order to compute time derivatives of f_{ij} at $t=0$, we will write

$$f_{ij}(t)=\langle\phi_i|e^{i(E_i-H)t}|\Psi_j(0)\rangle, \tag{2.16}$$

which yields at once

$$f_{ij}^{(n)}(0)=i^n\langle\phi_i|(E_i-H)^n|\Psi_j(0)\rangle. \tag{2.17}$$

For the transition rate we need only

$$\dot{f}_{ij}(0)=-iR_{ij}(\epsilon). \tag{2.18}$$

From (2.13) and (2.18) we have

$$\left[\frac{d}{dt}|f_{ij}(t)|^2\right]_{t=0}=2\delta_{ij}\,\mathrm{Im}R_{jj}(\epsilon)+\frac{2\epsilon}{(E_j-E_i)^2+\epsilon^2}|R_{ij}(\epsilon)|^2. \tag{2.19}$$

We are now in a position to deal with the normalization N_j of the state vector. Since the ϕ_j are a complete set of states,

$$\sum_i|f_{ij}|^2=N_j. \tag{2.20}$$

Using (2.19) and (2.20), and the fact that N_j is constant in time, we see that

$$2\,\mathrm{Im}R_{jj}(\epsilon)+\sum_i\frac{2\epsilon}{(E_j-E_i)^2+\epsilon^2}|R_{ij}(\epsilon)|^2=0. \tag{2.21}$$

If we now compute N_j from (2.13) and (2.20), we obtain

$$N_j=1+\frac{2}{\epsilon}\,\mathrm{Im}R_{jj}(\epsilon)+\sum_i\frac{1}{(E_j-E_i)^2+\epsilon^2}|R_{ij}(\epsilon)|^2, \tag{2.22}$$

and, simplifying with the use of (2.21),

$$N_j=1+(1/\epsilon)\,\mathrm{Im}R_{jj}(\epsilon). \tag{2.23}$$

We may remark that $R_{jj}(\epsilon)\sim L^{-3}$ and thus the double limiting process makes N_j tend to 1.

Now the differential cross section for the transition $j\to i$ is equal to the transition rate divided by the flux vL^{-3}, where v is the relative velocity of the colliding systems. Except for the single state j, to be discussed afterward, we have for each i the following expression for the differential cross section, using (2.19):

$$\sigma_{ij}=\lim_{\substack{\epsilon\to0^+\\L\to\infty}}\frac{2\epsilon}{(E_j-E_i)^2+\epsilon^2}|R_{ij}(\epsilon)|^2L^3v^{-1}. \tag{2.24}$$

Now the factor in parentheses tends to $2\pi\delta(E_j-E_i)$ which is to be interpreted as 2π times the density in energy of final states i at energy E_j; the conservation of energy is understood as well. If the volume in momentum space per unit energy about state i is ω_i, then the density of final states is $L^3\omega_i$ and we have

$$\sigma_{ij}=2\pi|\mathcal{R}_{ij}|^2\omega_iv^{-1}. \tag{2.25}$$

It is clear that the single state j considered as a possible final state will contribute nothing to the density of final states; nevertheless, the rate of change of w_{jj} is of importance when considered as the rate of depletion of the initial state.

From the fact that the normalization is preserved in time it is clear that the rate of decrease of w_{jj} is given precisely by the rate of transition into all other states $i\neq j$. If we set $i=j$ in Eq. (2.19) we obviously obtain

$$N_j\frac{d}{dt}w_{jj}\Big]_{t=0}=2\,\mathrm{Im}R_{jj}(\epsilon)+\frac{2}{\epsilon}|R_{jj}(\epsilon)|^2. \tag{2.26}$$

Now in the limit of $L\to\infty$ (and thus $\epsilon\to0^+$) the second term is vanishingly small in comparison to the first term [see discussion following Eq. (2.15)]. We have thus deduced the well-known theorem (in the limit $\epsilon\to0^+$),

$$\begin{aligned}-2\,\mathrm{Im}\mathcal{R}_{jj}/L^3&=\frac{1}{L^3}\sum_{i\neq j}\frac{2\pi}{L^3}|\mathcal{R}_{ij}|^2\delta(E_j-E_i),\\ \sum_{i\neq j}\sigma_{ij}&=-\frac{2}{v}\,\mathrm{Im}\mathcal{R}_{jj}.\end{aligned} \tag{2.27}$$

So far we have made use only of $f_{ij}(0)$ and $\dot{f}_{ij}(0)$. An examination of higher derivatives is not necessary for the calculation of cross sections, but sheds some light on the meaning of our mathematical description of scattering. A discussion of the second derivative of w_{ij} suffices to illustrate the point. One finds easily that

$$\dot{N}_jw_{ij}(0)=\epsilon\left\{\delta_{ij}2\,\mathrm{Im}R_{jj}(\epsilon)+\frac{4\epsilon}{(E_j-E_i)^2+\epsilon^2}|R_{ij}(\epsilon)|^2\right\}+\frac{2\epsilon(E_i-E_j)\,\mathrm{Im}V_{ij}R_{ij}^*(\epsilon)}{(E_i-E_j)^2+\epsilon^2}-\frac{2\epsilon^2\,\mathrm{Re}V_{ij}R_{ij}^*(\epsilon)}{(E_i-E_j)^2+\epsilon^2}. \tag{2.28}$$

For any finite value of ϵ, this expression is a perfectly well behaved function of energy. Consequently, if one computes $\dot{w}_{ij}(t)$ approximately as

$$\dot{N}_j\dot{w}_{ij}(t)\approx\dot{N}_j[\dot{w}_{ij}(0)+t\ddot{w}_{ij}(0)+\cdots], \tag{2.29}$$

one sees by comparing (2.19) and (2.28) that the second term is of order ϵt compared to the first term. Thus for times less than ϵ^{-1}, $\dot{w}_{ij}(t)\approx\dot{w}_{ij}(0)$.

From the preceding discussion it has become clear that for practical purposes one may ignore the complications of the double limiting processes and deal with the state vector $\psi_j^{(+)}$ obtained by letting ϵ tend to 0, which evidently satisfies

$$\psi_j^{(+)}=\phi_j+G^{(+)}(E_j)V\psi_j^{(+)}. \tag{2.30}$$

Cross sections can be computed from the quantities

$$R_{ij}=\langle\phi_i|V|\psi_j^{(+)}\rangle \tag{2.31}$$

in an obvious way. It is evident from (2.28) that $\psi_j^{(+)}$ is an eigenstate of the total Hamiltonian with eigenvalue

E_j. It is the conventional stationary solution of the scattering problem corresponding to an "incoming wave" ϕ_j.

Although it is useful to work with $\psi_j^{(+)}$, it is necessary to exercise care on account of its singular character when expressed in terms of ϕ's. For example, the operator K is not Hermitian when it appears between ϕ_j and $\psi_j^{(+)}$, since they satisfy different boundary conditions at infinity. Rather,

$$\langle K\phi_j|\psi_j^{(+)}\rangle-\langle\phi_j|K\psi_j^{(+)}\rangle=\langle\phi_j|V|\psi_j^{(+)}\rangle. \quad (2.32)$$

For most purposes it is safe to consider the normalization $\langle\psi_j^{(+)}|\psi_j^{(+)}\rangle$ to be 1.

III. S MATRIX FORMALISM

In this section we shall outline the connection between the preceding discussion and the formalism of the interaction representation; the introduction of Heisenberg's S matrix[3] then follows in a natural way, as pointed out by Schwinger.[4] Our development will be similar to that of Lippmann and Schwinger,[1] though different in point of view.

Starting again with the Schrödinger equation (2.1), we remove, in the usual way, the time dependence of the state vector associated with K by a unitary transformation to the interaction representation. Let

$$\Psi'(t)=e^{iKt}\Psi(t). \quad (3.1)$$

According to our definition, the interaction representation reduces to the Schrödinger representation at time $t=0$. The new state vector satisfies

$$i\partial\Psi'(t)/\partial t=V(t)\Psi'(t), \quad (3.2)$$

where

$$V(t)=e^{iKt}Ve^{-iKt}. \quad (3.3)$$

We introduce the unitary operator $U(t, t_0)$ such that

$$\Psi'(t)=U(t, t_0)\Psi'(t_0) \quad (3.4)$$

for each solution $\Psi'(t)$ of (3.2). $U(t, t_0)$ has two obvious properties that should be noted:

$$U(t, t)=1, \quad (3.5)$$

$$U(t, t_0)=U(t, t')U(t', t_0). \quad (3.6)$$

Let us exhibit some explicit formulas for $U(t, t_0)$. First, we may use our knowledge of the time dependence of $\Psi(t)$:

$$\Psi(t)=e^{-iH(t-t_0)}\Psi(t_0). \quad (3.7)$$

From (3.1) and (3.4) it is then clear that

$$U(t, t_0)=e^{iKt}e^{-i(K+V)(t-t_0)}e^{-iKt_0}. \quad (3.8)$$

To express $U(t, t_0)$ in terms of quantities in the interaction representation, we differentiate (3.8) with respect to t, obtaining

$$i\partial U(t, t_0)/\partial t=e^{iKt}Ve^{-i(K+V)(t-t_0)}e^{-iKt_0}, \quad (3.9)$$

which we may rewrite with the aid of (3.3) and (3.8) as

$$i\partial U(t, t_0)/\partial t=V(t)U(t, t_0). \quad (3.10)$$

Integrating both sides of (3.10) from t_0 to t, we have

$$U(t, t_0)=1-i\int_{t_0}^{t}dt'V(t')U(t', t_0). \quad (3.11)$$

If we differentiate (3.8) with respect to t_0 and then integrate, we obtain

$$U(t, t_0)=1+i\int_{t}^{t_0}dt'U(t, t')V(t'). \quad (3.12)$$

The formal solutions of (3.11) and (3.12) by iteration can be written, with the aid of Dyson's ordering operation,[5] as

$$U(t, t_0)=\left(\exp\left[-i\int_{t_0}^{t}dt'V(t')\right]\right)_{+}, \quad (3.13)$$

and

$$U(t, t_0)=\left(\exp\left[i\int_{t}^{t_0}dt'V(t')\right]\right), \quad (3.14)$$

respectively, where the symbol $(\ )_+$ means that the terms in the power series development are to be ordered with the functions of earliest times standing to the right and $(\ )_-$ indicates ordering in the opposite sense.

It is customary to introduce operators such as $U(t, -\infty)$ by allowing t_0 to approach $-\infty$ in such equations as (3.11)–(3.14). That it is not completely straightforward to do so becomes clear if we try to substitute $t_0=-\infty$ into Eq. (3.8). However, we are faced with no great mystery. The integrals in (3.11)–(3.14) may be exemplified by the second term in the expansion of (3.13):

$$-i\int_{t_0}^{t}dt'V(t')=-i\int_{t_0}^{t}dt'e^{iKt'}Ve^{-iKt'}. \quad (3.15)$$

The limit as t_0 tends to $-\infty$ of a matrix element of this operator will exist only if the limit is defined in such a way that oscillatory terms are made to vanish at $-\infty$. But with respect to such a limiting process, the limit of Eq. (3.8) will have meaning as well, as we shall see. Moreover, the work of the preceding section has already provided us with a suitable limiting process.

Let us transform the state vector $\Psi_j^{(+)}(t)$ of Sec. II to the interaction representation using (3.1). We obtain

$$\Psi_j'^{(+)}(t)=e^{iKt}e^{-iHt}\int_{-\infty}^{0}dTe^{iTT}e^{iHT}e^{-iKT}\phi_j, \quad (3.16)$$

[3] J. A. Wheeler, Phys. Rev. 52, 1107 (1937); W. Heisenberg, Z. Physik 120, 513, 673 (1943).

[4] J. Schwinger, Phys. Rev. 74, 1439 (1948). See also reference 1.

[5] F. J. Dyson, Phys. Rev. 75, 486 (1949).

which we may write, using (3.8), as

$$\Psi_j^{(+)}(t)=\epsilon\int_{-\infty}^{0}dTe^{\epsilon T}U(t,T)\phi_j. \tag{3.17}$$

Now

$$\lim_{T\to-\infty}{}_L f(T)\equiv\lim_{\epsilon\to0}\epsilon\int_{-\infty}^{0}dTe^{\epsilon T}f(T) \tag{3.18}$$

is an example of the kind of limiting process we need. If f possesses a genuine limit as $T\to-\infty$, the L operation yields the same one; but if f oscillates as $T\to-\infty$, the L operation gives 0. So we will take

$$U(t,-\infty)=\lim_{\epsilon\to0^+}\epsilon\int_{-\infty}^{0}dTe^{\epsilon T}U(t,T). \tag{3.19}$$

In an analogous way, we define

$$U(\infty,t)=\lim_{\epsilon\to0^+}\epsilon\int_{0}^{\infty}dTe^{-\epsilon T}U(T,t), \tag{3.20}$$

etc. All the relations are now true that can be obtained by setting t or t_0 equal to $\pm\infty$ as in the integral equations of the interaction representation. One may show, for example, that Eq. (3.11) does have the limiting form

$$U(t,-\infty)=1-i\int_{-\infty}^{t}dt'V(t')U(t',-\infty). \tag{3.21}$$

It is well known that in the notation of the interaction representation Heisenberg's S matrix takes the form $U(\infty,-\infty)$. It is clearly a matter of indifference at this point whether we define S by applying the two limiting processes (3.19) and (3.20) to $U(t,t_0)$ or by applying to (3.21) any limit that will give the usual meaning to the oscillatory integrals, so as to obtain

$$U(\infty,-\infty)\equiv S=1-i\int_{-\infty}^{\infty}dt'V(t')U(t',-\infty). \tag{3.22}$$

We may now substantiate the claim that sensible results follow from allowing t_0 to tend to $-\infty$ in Eq. (3.8) according to the rule (3.19). If we use the completeness relation,

$$1=\sum_j\phi_j\rangle\langle\phi_j, \tag{3.23}$$

we find that $U(t,-\infty)$ can be expressed in the form

$$U(t,-\infty)=e^{iKt}e^{-i(K+V)t}\lim_{\epsilon\to0^+}\sum_j\frac{\epsilon}{\epsilon+i(H-E_j)}\phi_j\rangle\langle\phi_j. \tag{3.24}$$

In view of Eq. (2.8) and the discussion at the end of Sec. II, we have

$$U(0,-\infty)=\sum_j\psi_j^{(+)}\rangle\langle\phi_j, \tag{3.25}$$

or

$$U(0,-\infty)\phi_j=\psi_j^{(+)}. \tag{3.26}$$

The operator $U(0,-\infty)$, which we shall call $\Omega^{(+)}$, is clearly the one that forms the singular wave-function matrix of Møller:[6]

$$\langle\phi_i|\Omega^{(+)}|\phi_j\rangle=\langle\phi_i|\psi_j^{(+)}\rangle. \tag{3.27}$$

Acting on the state ϕ_j, it produces that eigenstate of the total Hamiltonian, corresponding to ϕ_j as an incident wave. Similarly the operator

$$U(0,+\infty)\equiv\Omega^{(-)} \tag{3.28}$$

carries ϕ_j into the eigenstate of the total Hamiltonian corresponding to ϕ_j as an outgoing wave:

$$\Omega^{(-)}\phi_j=\psi_j^{(-)}=\lim_{\epsilon\to0^+}\frac{\epsilon}{\epsilon-i(H-E_j)}\phi_j. \tag{3.29}$$

Instead of (2.28), we have for $\psi_j^{(-)}$ the equation

$$\psi_j^{(-)}=\phi_j+G^{(-)}(E_j)V\psi_j^{(-)}, \tag{3.30}$$

where

$$G^{(-)}(E_j)=\lim_{\epsilon\to0^+}\frac{1}{E_j-K-i\epsilon}. \tag{3.31}$$

Let us now establish the properties of the U matrices with infinite arguments that correspond to Eq. (3.6) for finite times. It can easily be seen from (3.6) and (3.20) that, for example,

$$U(\infty,t)=U(\infty,0)U(0,t), \tag{3.32}$$

and thus by (3.19)

$$U(\infty,-\infty)=U(\infty,0)U(0,-\infty). \tag{3.33}$$

Now $U(\infty,-\infty)$ is S and $U(0,-\infty)$ is $\Omega^{(+)}$, but $U(\infty,0)$ remains to be discussed.

Equations (3.5) and (3.6) tell us that

$$U(t,t_0)U(t_0,t)=U(t_0,t)U(t,t_0)=1. \tag{3.34}$$

Since the U's are unitary for finite times, we have

$$U(t,t_0)=U(t_0,t)^\dagger. \tag{3.35}$$

If we apply either of the limits (3.19) or (3.20) to the relation (3.35), it remains unchanged, and thus

$$U(\pm\infty,0)=U(0,\pm\infty)^\dagger. \tag{3.36}$$

So Eq. (3.33) may be rewritten

$$S=\Omega^{(-)\dagger}\Omega^{(+)}. \tag{3.37}$$

We must still investigate such products as $U(-\infty,0)U(0,-\infty)$. But Eqs. (3.23) and (3.36) yield at once

$$\begin{aligned}U(-\infty,0)U(0,-\infty)&=\Omega^{(+)\dagger}\Omega^{(+)}\\&=\sum_{i,j}\phi_i\rangle\langle\psi_i^{(+)}|\psi_j^{(+)}\rangle\langle\phi_j\\&=\sum_i\phi_i\rangle\langle\phi_i=1\end{aligned} \tag{3.38}$$

[6] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 23, No. 1 (1945).

with the use of the limiting normalization to unity of the $\psi_j^{(+)}$. Similarly,

$$U(\infty,0)U(0,\infty)=\Omega^{(-)\dagger}\Omega^{(-)}=1. \tag{3.39}$$

However, the Ω's are not necessarily unitary, since

$$\Omega^{(\pm)}\Omega^{(\pm)\dagger}=\sum_i \psi_i^{(\pm)}\rangle\langle\psi_i^{(\pm)} \tag{3.40}$$

is not necessarily 1. If there are bound states among the eigenstates of H, that is, states of energy less than that of any eigenstate of K with the same symmetry quantum numbers, then the $\psi_i^{(\pm)}$ are not a complete set, and (3.40) may be restated as

$$\Omega^{(\pm)}\Omega^{(\pm)\dagger}=1-\sum_a \psi_a\rangle\langle\psi_a, \tag{3.41}$$

where the ψ_a are the bound states, satisfying

$$H\psi_a=E_a\psi_a. \tag{3.42}$$

In order to verify directly what is implicit in the preceding equations, namely,

$$\Omega^{(\pm)\dagger}\psi_a=0, \tag{3.43}$$

as well as that

$$\Omega^{(\pm)\dagger}\psi_i^{(\pm)}=\phi_i, \tag{3.44}$$

let us use (3.16), (3.19), and (3.20), which yield

$$\Omega^{(\pm)\dagger}\psi=\lim_{\epsilon\to 0^+}\frac{\epsilon}{\epsilon\pm i(K-E)}\psi, \tag{3.45}$$

where ψ is any eigenfunction of H with eigenvalue E. For a bound state, $K-E$ can never vanish, since K has no eigenvalues as low as E; hence the relation (3.43). For a state in the continuum, we have

$$\lim_{\epsilon\to 0^+}\frac{\epsilon}{\epsilon\pm i(K-E)}\psi^{(\pm)}=(1-G^{(\pm)}(E)V)\psi^{(\pm)}=\phi \tag{3.46}$$

by Eqs. (2.28) and (3.30); hence the relation (3.44).

The matrix elements of the S matrix may be computed as follows: From (3.24) and (3.25) we find that

$$U(t,-\infty)=e^{iKt}\sum_j e^{-iE_jt}\psi_j^{(+)}\rangle\langle\phi_j, \tag{3.47}$$

and so

$$V(t)U(t,-\infty)=\sum_{i,j}\phi_i\rangle\langle\phi_i|e^{i(E_i-E_j)t}V\psi_j^{(+)}\rangle\langle\phi_j. \tag{3.48}$$

Substituting into (3.22), we have

$$S=1-\sum_{i,j}\phi_i\rangle 2\pi i\delta(E_i-E_j)R_{ij}\langle\phi_j, \tag{3.49}$$

or

$$S_{ij}=\langle\phi_i|S|\phi_j\rangle=\delta_{ij}-2\pi i\delta(E_i-E_j)R_{ij}. \tag{3.50}$$

Another interesting form is provided by substituting Eq. (3.39) into the definition of S_{ij}:

$$\begin{aligned}S_{ij}&=\langle\phi_i|\Omega^{(-)\dagger}\Omega^{(+)}|\phi_j\rangle\\&=\langle\Omega^{(-)}\phi_i|\Omega^{(+)}\phi_j\rangle=\langle\psi_i^{(-)}|\psi_j^{(+)}\rangle.\end{aligned} \tag{3.51}$$

Our final task is to prove that the S matrix is unitary. This may be done in a variety of ways. For example, Eq. (2.21) and the analogous equation involving the incoming wave state vectors $\psi^{(-)}$ may be made the basis of such a proof.[6] We shall proceed from the connection between the S matrix and the $\Omega^{(\pm)}$ operators [Eq. (3.39)]. We have

$$\begin{aligned}S^\dagger S&=\Omega^{(+)\dagger}\Omega^{(-)}\Omega^{(-)\dagger}\Omega^{(+)}\\&=\Omega^{(+)\dagger}[1-\textstyle\sum_a\psi_a\rangle\langle\psi_a]\Omega^{(+)}\\&=1-\Omega^{(+)\dagger}\textstyle\sum_a\psi_a\rangle\langle\psi_a\Omega^{(+)}.\end{aligned} \tag{3.52}$$

The second term is effectively zero, since $\Omega^{(+)}$ operating on a state ϕ_j produces one of the states $\psi_j^{(+)}$ [according to Eq. (3.27)] which is orthogonal to the bound states ψ_a. Thus we have completed one-half of the proof:

$$S^\dagger S=1. \tag{3.53}$$

The remainder of the proof is as follows:

$$\begin{aligned}SS^\dagger&=\Omega^{(-)}\Omega^{(+)}\Omega^{(+)\dagger}\Omega^{(-)}\\&=1-\Omega^{(-)\dagger}\textstyle\sum_a\psi_a\rangle\langle\psi_a\Omega^{(-)}=1,\end{aligned} \tag{3.54}$$

since $\Omega^{(-)}$ produces a state $\psi^{(-)}$ which is orthogonal to the ψ_a.[7]

IV. EXAMPLES

Our first example is a discussion of a scattering process in which there are two potentials acting. Problems which fall into this category are the scattering of particles under the combined influence of Coulomb and nuclear forces (the actual case of the Coulomb field requires a detailed discussion of phase factors, etc., which may be carried out explicitly and which, in fact, yields results identical with those to be discussed below[8]); bremsstrahlung, where one has a Coulomb field as well as interaction with the radiation field; the analogous problem of meson production in nucleon-nucleon collisions; the photoelectric effect; etc. Problems involving two potentials to which the impulse approximation is applied are best discussed in a slightly different way.[9] The motivation for our approach is evidently that it may be advantageous to treat one of the potentials exactly and the other approximately; furthermore, as will be mentioned below, the nature of the physical question being asked introduces a possible ambiguity into the mathematics.

At first sight the whole discussion might appear to be trivial: One would merely replace the basic set of states ϕ, introduced earlier, by a set of states χ which are eigenfunctions of $K+U$, where U is the part of the potential that is to be treated exactly. Cross sections would then be obtained from expressions of the form

$$\langle\chi|V|\psi^{(+)}\rangle,$$

[7] After this manuscript was completed, a paper by S. T. Ma appeared [S. T. Ma, Phys. Rev. 87, 652 (1952)], which contains some of the results discussed in the latter part of this section. It was, nevertheless, felt desirable to include them in the interest of completeness.

[8] J. B. French and M. L. Goldberger, Phys. Rev. 87, 899 (1952).

[9] G. F. Chew and M. L. Goldberger, Phys. Rev. 87, 778 (1952).

where V is the additional potential. Let us see to what extent such a procedure is legitimate.

The difficulty arises from the fact that the cross sections for the processes mentioned above are classified in terms of truly noninteracting final states. The true state vector is defined by the integral equation

$$\psi_a^{(+)}=\phi_a+\frac{1}{E-K+i\epsilon}(U+V)\psi_a^{(+)}, \tag{4.1}$$

where ϕ_a is an eigenstate of K belonging to energy E. (The common energy of initial and final states will be called E.) The probability of transition to another plane wave state ϕ_b is, as we have seen in Sec. II, proportional to the absolute square of

$$R_{ba}=\langle\phi_b|R|\phi_a\rangle=\langle\phi_b|U+V|\psi_a^{(+)}\rangle. \tag{4.2}$$

This is clearly the quantity of physical interest, the transition rate into a *true plane wave state*, ϕ_b. We shall see that in many cases the expression for R_{ba} in Eq. (4.2) can indeed be written in the form $\langle\chi|V|\psi^+\rangle$ alluded to above, but *only with a suitable choice of boundary conditions on* χ.[10]

Let us introduce state vectors analogous to the $\psi_a^{(-)}$ used previously: These are the solutions $\chi_b^{(-)}$ of the problem with $V=0$ defined by

$$\chi_b^{(-)}=\phi_b+\frac{1}{E-K-i\epsilon}U\chi_b^{(-)}. \tag{4.3}$$

Substituting for ϕ_b in (4.2) and using (4.1), we find that

$$R_{ba}=\langle\chi_b^{(-)}|V|\chi_a^{(+)}\rangle+\langle\chi_b^{(-)}|U|\phi_a\rangle. \tag{4.4}$$

It is very easy to show[1] that

$$\langle\chi_b^{(-)}|U|\phi_a\rangle=\langle\phi_b|U|\chi_a^{(+)}\rangle, \tag{4.5}$$

where $\chi_a^{(+)}$ is defined by

$$\chi_a^{(+)}=\phi_a+\frac{1}{E-K+i\epsilon}U\chi_a^{(+)}. \tag{4.6}$$

The second term in (4.4) is thus simply the scattering amplitude which would be found even if V were zero. In many applications this term is zero; for example, in the case of bremsstrahlung there is a photon in the final stat and consequently the matrix element will vanish. It does *not* vanish in the case of combined Coulomb and nuclear scattering. The first term shows that the famous incoming wave solution[11] $\chi_b^{(-)}$, which has frequently been a source of confusion, appears quite naturally.

[10] K. Watson, Phys. Rev. 88, 1163 (1952). The result expressed in (4.4) below has also been obtained by Watson (without the second term). His derivation is quite similar to ours.

[11] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1933), Chap. VI; A. Sommerfeld, *Wellenmechanik* (Fredrick Unger Publishing Company, New York, 1947), p. 457.

The very implicit dependence of $\psi_a^{(+)}$ on U may be made more explicit by remarking that $\psi_a^{(+)}$ also satisfies the equation

$$\psi_a^{(+)}=\chi_a^{(+)}+\frac{1}{E-K-U+i\epsilon}V\psi_a^{(+)}. \tag{4.7}$$

Since it is important to realize that the original boundary conditions on $\psi_a^{(+)}$ introduced in (4.1) are being maintained effectively in (4.7) we shall derive this result. We write

$$\begin{aligned}\psi_a^{(+)}&=\phi_a+\frac{1}{E-K-U-V+i\epsilon}(U+V)\phi_a,\\ \chi_a^{(+)}&=\phi_a+\frac{1}{E-K-U+i\epsilon}U\phi_a,\end{aligned} \tag{4.8}$$

as in Eq. (2.10). Subtracting, we have, after some manipulation,

$$\psi_a^{(+)}=\chi_a^{(+)}+\frac{1}{E-K-U-V+i\epsilon}V\chi_a^{(+)}, \tag{4.9}$$

which is the solution of (4.7). In the case where V is small, Eq. (4.4) becomes approximately

$$R_{ba}\simeq\langle\chi_b^{(-)}|V|\chi_a^{(+)}\rangle+\langle\phi_b|U|\chi_a^{(+)}\rangle. \tag{4.10}$$

Equation (4.10) takes the potential U into account exactly as long as one is content with first-order accuracy in V. It is to be noticed that to second order in V there is an additional U dependence, since

$$\psi_a^{(+)}\simeq\chi_a^{(+)}+\frac{1}{E-K-U+i\epsilon}V\chi_a^{(+)}+\cdots. \tag{4.11}$$

In the case of the photoelectric effect or the process $\pi^++d\rightarrow 2p$ one is confronted with a slightly new problem. One is dealing with an initial state which, although it lies in the continuum, is essentially a bound state. Let us proceed from first principles and compute the transition probability directly. We shall keep in mind, for the sake of terminology, the process $\pi^++d\rightarrow 2p$. The physics of the problem leads us to the state vector $\Psi_a^{(+)}$ defined by

$$\Psi_a^{(+)}=\psi_0+\frac{1}{E_a-K-U+i\epsilon}V\Psi_a^{(+)}, \tag{4.12}$$

where ψ_0 represents the product of a plane wave meson state vector and a bound deuteron state vector. ψ_0 satisfies the integral equation

$$\psi_0=\frac{1}{E_a-K+i\epsilon}U\psi_0, \tag{4.13}$$

where U is the deuteron potential. The meson coordinates are contained in a completely trivial way in this equation since there is no interaction. It is convenient, however, not to make an explicit separation. We now

ask for the rate of transition into a plane wave state in the usual way:

$$N_a\dot{w}_{ba}=\frac{\partial}{\partial t}|\langle\phi_b|e^{i(E_b-K-U-V)t}|\Psi_a^{(+)}\rangle|^2. \quad (4.14)$$

(We must, of course, keep ϵ finite in order that the necessary partial integrations can be carried out.) We shall evaluate (4.14) at $t=0$. As in Sec. II, we obtain

$$N_a\dot{w}_{ba}=-i\langle\phi_b|U+V|\Psi_a^{(+)}\rangle\langle\phi_b|\Psi_a^{(+)}\rangle^*+\text{c.c.} \quad (4.15)$$

The evaluation of $\langle\phi_b|\Psi_a^{(+)}\rangle$ proceeds as follows: First write $\Psi_a^{(+)}$ in the form

$$\Psi_a^{(+)}=\psi_0+\frac{1}{E_a-K-U-V+i\epsilon}V\psi_0. \quad (4.16)$$

Then

$$\begin{aligned}\langle\phi_b|\Psi_a^{(+)}\rangle&=\langle\phi_b|\psi_0\rangle+\left\langle\phi_b\left|\frac{1}{E_a-K-U-V+i\epsilon}V\right|\psi_0\right\rangle\\&=\langle\phi_b|\psi_0\rangle+\left\langle\phi_b\left|\frac{1}{E_a-K+i\epsilon}V\right|\psi_0\right\rangle\\&\quad+\left\langle\phi_b\left|\frac{1}{E_a-K+i\epsilon}(U+V)\right.\right.\\&\quad\left.\left.\times\frac{1}{E_a-K-U-V+i\epsilon}V\right|\psi_0\right\rangle\\&=\langle\phi_b|\psi_0\rangle+\left\langle\phi_b\left|\frac{1}{E_a-K+i\epsilon}V\right|\psi_0\right\rangle\\&\quad+\left\langle\phi_b\frac{1}{E_a-K+i\epsilon}(U+V)|\Psi_a^{(+)}-\psi_0\right\rangle\\&=\left\langle\phi_b\left|\psi_0-\frac{1}{E-K+i\epsilon}U\psi_0\right\rangle\right.\\&\quad+\left\langle\phi_b\left|\frac{1}{E_a-K+i\epsilon}(U+V)\Psi_a^{(+)}\right\rangle\right.\\&=\frac{1}{E_a-E_b+i\epsilon}\langle\phi_b|U+V|\Psi_a^{(+)}\rangle. \quad (4.17)\end{aligned}$$

Proceeding in the now familiar way, we deduce

$$\dot{w}_{ba}=2\pi|\langle\phi_b|U+V|\psi_a^{(+)}\rangle|^2\delta(E_a-E_b), \quad (4.18)$$

where we now imagine the limit $\epsilon\to 0^+$ and hence $\Psi_a^{(+)}\to\psi_a^{(+)}$. Now we shall transcribe this matrix element in a manner similar to that used in connection with (4.2). This time, however, we substitute for ϕ_b from the equation

$$\chi_b^{(-)}=\phi_b+\frac{1}{E-K-U-i\epsilon}U\phi_b. \quad (4.19)$$

The common energy of ϕ_b and $\psi_a^{(+)}$ is again denoted by E. We write

$$\begin{aligned}R_{ba}&=\langle\phi_b|U+V|\psi_a^{(+)}\rangle\\&=\langle\phi_b|U|\psi_a^{(+)}\rangle+\langle\chi_b^{(-)}|V|\psi_a^{(+)}\rangle\\&\quad-\left\langle\frac{1}{E-K-U-i\epsilon}U\phi_b|V|\psi_a^{(+)}\right\rangle\\&=\langle\phi_b|U|\psi_a^{(+)}\rangle+\langle\chi_b^{(-)}|V|\psi_a^{(+)}\rangle\\&\quad-\left\langle\phi_b\left|U\frac{1}{E-K-U+i\epsilon}V\right|\psi_a^{(+)}\right\rangle\\&=\langle\phi_b|X|\psi_a^{(+)}\rangle+\langle\chi_b^{(-)}|V|\psi_a^{(+)}\rangle\\&\quad-\langle\phi_b|U|\psi_a^{(+)}-\psi_0\rangle.\end{aligned}$$

Finally

$$R_{ba}=\langle\chi_b^{(-)}|V|\psi_a^{(+)}\rangle+\langle\phi_b|U|\psi_0\rangle. \quad (4.20)$$

In the class of examples being considered, where ψ_0 is the product of a plane wave state and a bound state, the second term vanishes. Hence the transition probability per unit time becomes

$$\dot{w}_{ba}=2\pi|\langle\chi_b^{(-)}|V|\psi_a^{(+)}\rangle|^2\delta(E_a-E_b). \quad (4.21)$$

It is perhaps worth noting that Eq. (4.21) as well as the analogous one, for an initial continuum, contained in (4.4), *would not be correct* with $\chi_b^{(+)}$ written in place of $\chi_b^{(-)}$.

As another example of our formalism, we shall present a rigorous theory of the so-called "pick-up process."[12] The general category covered by this example is considerably broader in that it applies to rearrangement collisions quite generally. For definiteness, we consider the following idealized problem: A proton is bound to a fixed scattering center by a potential U and is bombarded by neutrons with energy E_i which interact with the proton through the potential V. The neutron and proton may be bound together by V to form a deuteron and we wish to compute the transition probability per unit time for producing deuterons.

We introduce the state vectors ψ_0 and ψ_f, which satisfy the equations

$$(K+U)\psi_0=E_i\psi_0, \quad (K+V)\psi_f=E_f\psi_f, \quad (4.22)$$

where ψ_0 represents the initially bound proton and incident neutron and ψ_f, the deuteron (with its center of gravity motion). $\Psi^{(+)}$, the complete state vector of the system, can be seen from the physical boundary conditions to be the solution of the integral equation

$$\Psi^{(+)}=\psi_0+\frac{1}{E_i-K-U+i\epsilon}V\Psi^{(+)}, \quad (4.23)$$

[12] G. F. Chew and M. L. Goldberger, Phys. Rev. 77, 899 (1950).

namely,

$$\Psi^{(+)}=\psi_0+\frac{1}{E_i-K-U-V+i\epsilon}V\psi_0. \quad (4.24)$$

The transition probability of interest is given by

$$N_0\dot{w}_{f0}=\frac{\partial}{\partial t}|\langle\psi_f e^{i(E_f-K-U-V)t}|\psi^{(+)}\rangle|^2, \quad (4.25)$$

which we again compute at $t=0$. We find that

$$N_0\dot{w}_{f0}=-i\langle\psi_f|U|\Psi^{(+)}\rangle\langle\psi_f|\Psi^{(+)}\rangle^*+\text{c.c.} \quad (4.26)$$

Now

$$\begin{aligned}\langle\psi_f|\Psi^{(+)}\rangle&=\langle\psi_f|\psi_0\rangle+\left\langle\psi_f\left|\frac{1}{E_i-K-U-V+i\epsilon}V\right|\psi_0\right\rangle\\&=\langle\psi_f|\psi_0\rangle+\left\langle\psi_f\left|\frac{1}{E_i-K-V+i\epsilon}V\right|\psi_0\right\rangle\\&\quad+\left\langle\psi_f\left|\frac{1}{E_i-K-U+i\epsilon}U\frac{1}{E-K-U-V+i\epsilon}V\right|\psi_0\right\rangle\\&=\langle\psi_f|\psi_0\rangle+\frac{1}{E_i-E_f+i\epsilon}\{\langle\psi_f|V|\psi_0\rangle\\&\qquad+\langle\psi_f|U|\Psi^{(+)}\rangle-\langle\psi_f|U|\psi_0\rangle\}. \quad (4.27)\end{aligned}$$

From Eqs. (4.22) we find that

$$\begin{aligned}&\langle\psi_f|V|\psi_0\rangle-\langle\psi_f|U|\psi_0\rangle\\&\quad=(E_f-E_i)\langle\psi_f|\psi_0\rangle-\{\langle K\psi_f|\psi_0\rangle-\langle\psi_f|K|\psi_0\rangle\}. \quad (4.28)\end{aligned}$$

The second term in (4.26) reduces to a surface integral which vanishes in the limit of infinite quantization volume. Thus

$$\langle\psi_f|\Psi^{(+)}\rangle=\frac{1}{E_i-E_f+i\epsilon}\langle\psi_f|U|\Psi^{(+)}\rangle. \quad (4.29)$$

We find immediately that

$$\dot{w}_{f0}=2\pi|\langle\psi_f|U|\psi^{(+)}\rangle|^2\delta(E_i-E_f), \quad (4.30)$$

in the limit $\epsilon\to 0^+$, $\Psi^{(+)}\to\psi^{(+)}$. We shall not discuss various methods of approximation which have been developed to evaluate (4.30); this will be taken up in a separate paper by one of us (M.L.G.) in collaboration with Chew.[13]

The form of Eq. (4.30) would seem to be somewhat surprising, in that it is not at all the result one would intuitively write down. The interaction V between the projectile and the bound particle is buried in a complicated way. It is not difficult, however, to deduce from (4.30) a more natural looking result. Calling the common energy E, and substituting the explicit form for $\psi^{(+)}$, we have

$$\begin{aligned}&\langle\psi_f|U|\psi^{(+)}\rangle\\&\quad=\langle\psi_f|U|\psi_0\rangle+\left\langle\psi_f\left|U\frac{1}{E-K-U-V-i\epsilon}V\right|\psi_0\right\rangle\\&\quad=\langle\psi_f|U|\psi_0\rangle\\&\qquad+\left\langle\frac{1}{E-K-U-V-i\epsilon}U\psi_f|V|\psi_0\right\rangle. \quad (4.31)\end{aligned}$$

We now define a new state vector $\psi^{(-)}$ which is the solution of

$$\psi^{(-)}=\psi_f+\frac{1}{E-K-V-i\epsilon}U\psi^{(-)}, \quad (4.32)$$

namely,

$$\psi^{(-)}=\psi_f+\frac{1}{E-K-U-V-i\epsilon}U\psi_f. \quad (4.33)$$

Thus (4.31) may be written, using (4.33) in the second term, as

$$\begin{aligned}\langle\psi_f|U|\psi^{(+)}\rangle&=\langle\psi^{(-)}|V|\psi_0\rangle+\langle\psi_f|U|\psi_0\rangle\\&\quad-\langle\psi_f|V|\psi_0\rangle. \quad (4.34)\end{aligned}$$

The last two terms lead to the surface integral discussed after Eq. (4.28) and may be dropped. Hence we have proved that

$$\langle\psi_f|U|\psi^{(+)}\rangle=\langle\psi^{(-)}|V|\psi^0\rangle. \quad (4.35)$$

This new form for the transition matrix element is the one which one would guess for the result. The reciprocity relationship expressed by (4.35) is the analog of a similar one quoted in Eq. (4.5).

V. SELF-ENERGIES

So far we have restricted ourselves to the consideration of Hamiltonians in which the interaction V induces a negligibly small shift of the energy levels in the continuous spectrum. But in order to discuss, for example, a theory of quantized fields, we must deal with the question of self-energies that are not infinitesimal. (The fact that for elementary particles without extension they often turn out to be infinite is without significance for our treatment; we may keep in mind, as an example of a finite theory, that of electrons and phonons in a lattice.)

Let us suppose, then, that the eigenvalues of the total Hamiltonian H are E_n while those of the portion K of the Hamiltonian that we have chosen to call "free" or "unperturbed" are δ_n; that is,

$$(K+V)\psi_n=E_n\psi_n \quad (5.1)$$

and

$$K\phi_n=\delta_n\phi_n. \quad (5.2)$$

For simplicity, we will assume that there are no bound states in either case.

[13] Equation (4.30) has also been derived independently by G. F. Chew and by G. C. Wick (private communication).

Now it is essential to note that the energy of the colliding systems, even at infinite separation, is *not* given by the $\mathcal{E}_n$ of the "initial" state, but by E_n. (For instance, the energy of two distant electrons at rest is not the sum of their mechanical masses, but of their total masses, and this includes a contribution from the interaction with the electromagnetic field.) Now in order to describe the scattering process correctly, one must assign to the incident train its correct frequency, and thus K must be modified formally so as to make its energy spectrum coincide with that of $K+V$ before it can be used as in Sec. II.

If Δ_n is defined by

$$\Delta_n = E_n - \mathcal{E}_n, \tag{5.3}$$

and the operator Δ by

$$\Delta = \sum_n \phi_n \rangle \Delta_n \langle \phi_n, \tag{5.4}$$

then

$$(K+\Delta)\phi_n = E_n \phi_n. \tag{5.5}$$

We may now write

$$H = (K+\Delta) + (V-\Delta), \tag{5.6}$$

and apply the methods of Sec. II in full, since the new interaction $(V-\Delta)$ produces no energy shift.

In place of Eq. (2.11) we have

$$\Psi_j^{(+)}(0) = \phi_j + \frac{1}{E_j - (K+\Delta) + i\epsilon}(V-\Delta)\Psi_j^{(+)}(0), \tag{5.7}$$

and in place of (2.10),

$$\Psi_j^{(+)}(0) = \phi_j + \frac{1}{E_j - K - V + i\epsilon}(V-\Delta)\phi_j. \tag{5.8}$$

In virtue of (5.3) and (5.4), we may write (5.8) in the form

$$\Psi_j^{(+)}(0) = \phi_j + \frac{1}{\mathcal{E}_j + \Delta_j - K - V + i\epsilon}(V-\Delta_j)\phi_j. \tag{5.9}$$

It is clear, then, that the computation of Ψ_j requires a knowledge only of Δ_j and not of the energy shifts of the other states.

The question of the determination of Δ_j has been discussed by Pirenne;[14] later he has shown[15] that his approach is fully equivalent, in the case of quantum electrodynamics, to the covariant mass renormalization procedure usually adopted in recent years. We will base our remarks on his ideas.

We note first that the R matrix element from which cross sections are computed, is, by analogy with Eq. (2.14), given by

$$\begin{aligned} R_{ij}(\epsilon) &= \langle \phi_i | V - \Delta | \Psi_j^{(+)}(0) \rangle \\ &= \langle \phi_i | V - \Delta_i | \Psi_j^{(+)}(0) \rangle, \end{aligned} \tag{5.10}$$

or, using (5.9),

$$\begin{aligned} R_{ij}(\epsilon) = {} & \langle \phi_i | V | \phi_j \rangle \\ & + \left\langle \phi_i \middle| (V-\Delta_i) \frac{1}{E_j + \Delta_j - K - V + i\epsilon} (V-\Delta_j) \middle| \phi_j \right\rangle \\ & - \Delta_i \delta_{ij}. \end{aligned} \tag{5.11}$$

Now we know that as ϵ^{-1} and L tend to infinity, $R_{ij}(\epsilon)$ must tend to 0 since otherwise the cross section would be infinite, or at least dependent on the normalization volume. Hence for an interaction V that produces level shifts which are not infinitesimal, the term $\Delta_i \delta_{ij}$ must be canceled by a portion of the other two terms on the right-hand side of (5.11). Thus the expression,

$$\begin{aligned} F_{ij} = {} & \langle \phi_i | w | \phi_j \rangle + \left\langle \phi_i \middle| (V-\Delta_i) \right. \\ & \left. \times \frac{1}{E_j + \Delta_j - K - V + i\epsilon} (V-\Delta_j) \middle| \phi_j \right\rangle, \end{aligned} \tag{5.12}$$

must be of the form of an infinitesimal plus a term proportional to δ_{ij} that does not vanish as $\epsilon \to 0$ and L tend to infinity. It is the latter term that Pirenne refers to as a "singularity." If we use the symbol $W(F_{ij})$ to mean the "singularity" in F_{ij} at $i=j$ then the self-energy as determined by the relation

$$\begin{aligned} \Delta_j = {} & \lim_{\epsilon \to 0^+} W \left\{ \langle \phi_j | V | \phi_j \rangle + \left\langle \phi_j \middle| (V-\Delta_j) \right. \right. \\ & \left. \left. \times \frac{1}{\mathcal{E}_j + \Delta_j - K - V + i\epsilon} (V-\Delta_j) \middle| \phi_j \right\rangle \right\}. \end{aligned} \tag{5.13}$$

Equation (5.13) may be solved for Δ_j by the use of perturbation theory in V or by other means if they are available.

In order to exhibit the reality of Δ_j, which is certainly not apparent from (5.13), let us rewrite Eq. (5.11) in the form

$$R_{ij}(\epsilon) = \langle \phi_i | V | \phi_j \rangle + \sum_k \frac{R_{ik} R_{jk}^*}{E_j - E_k + i\epsilon} - \Delta_j \delta_{ij}. \tag{5.14}$$

Taking the diagonal element and allowing ϵ to approach 0, we have

$$\begin{aligned} R_{jj} + i\pi \sum_k |R_{jk}|^2 \delta(E_j - E_k) \\ = \langle \phi_j | V | \phi_j \rangle + \mathrm{P} \sum_k \frac{|R_{jk}|^2}{E_j - E_k} - \Delta_j, \end{aligned} \tag{5.15}$$

where P means principal value. But by analogy to

[14] Jean Pirenne, Helv. Phys. Acta 21, 226 (1948).
[15] Jean Pirenne, Phys. Rev. 86, 395 (1952).

Eq. (2.21), we can show that

$$2\,\mathrm{Im}R_{jj}=-2\pi\sum_k |R_{jk}|^2\delta(E_j-E_k), \quad (5.16)$$

and hence (5.15) becomes

$$\mathrm{Re}R_{jj}+\Delta_j=\langle\phi_j|V|\phi_j\rangle+\mathrm{P}\sum_k\frac{|R_{jk}|^2}{E_j-E_k}. \quad (5.17)$$

The "singularity" in the right-hand side of (5.17) is the part that does not vanish for infinite normalizing volume and is equal to Δ_j. The remainder is infinitesimal and equal to $\mathrm{Re}R_{jj}$. Thus in place of (5.13) we may write

$$\Delta_j=W\left\{\langle\phi_j|V|\phi_j\rangle+\left\langle\phi_j\left|(V-\Delta_j)\frac{\mathrm{P}}{\mathcal{E}_j+\Delta_j-K-V}(V-\Delta_j)\right|\phi_j\right\rangle\right\}. \quad (5.18)$$

It is now clear how to compute any quantity of physical interest in the case where there are self-energies. If the transition to a covariant formalism is made, our remarks remain pertinent.

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The Energy-Momentum Tensor of the Electromagnetic Field inside Matter

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Two different energy-momentum tensors have been proposed to describe the electromagnetic field inside matter. Abraham suggested a symmetric tensor while Minkowski's tensor is nonsymmetric. With the aid of a thought experiment it is shown here that only the symmetric tensor satisfies the momentum conservation and center-of-mass theorems simultaneously.

THERE is an interesting point in the electrodynamics of moving media which is not yet satisfactorily settled. This problem is the following. The well-known connections in vacuum between the electric and magnetic field vectors E and H, the electromagnetic energy flux S, and the momentum density g are $\mathbf{S}=c(\mathbf{E}\times\mathbf{H})$, $\mathbf{g}=(1/c)(\mathbf{E}\times\mathbf{H})$. From these expressions it follows that the energy-momentum tensor $T_{ik}(i, k=1, 2, 3, 4)$ of the field in vacuum is symmetrical, since the space part is symmetrical and the time parts are simply the energy flux and momentum density, respectively, $T_{4\rho}=(1/c)\mathbf{S}$, $T_{\rho 4}=c\mathbf{g}(\rho=1, 2, 3)$, which are equal in view of the above expressions. Two different expressions have, however, been suggested for S and g and so for T_{ik} when the electromagnetic phenomena take place in matter. In both cases the space part of the energy-momentum tensor is the conventional Maxwell stress tensor. For S and g Abraham has retained the expressions valid for fields in vacuum[1] and so he obtains a symmetric energy-momentum tensor. Minkowski, however, proposed $\mathbf{S}=c(\mathbf{E}\times\mathbf{H})$, $\mathbf{g}=(1/c)(\mathbf{D}\times\mathbf{B})$; $\mathbf{D}=\epsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$, which entails a nonsymmetric energy-momentum tensor. For a long time Abraham's suggestion (the symmetric tensor) was commonly accepted, but quite recently von Laue[1] has shown that only Minkowski's assumption leads to a ray velocity (velocity of energy propagation) transforming like the particle velocity. This was considered as a weighty argument in favor of the nonsymmetrical energy-momentum tensor.

The aim of this note is to show by means of a very simple thought experiment that only the symmetric energy-momentum tensor satisfies simultaneously the momentum conservation and center-of-mass theorems. (We mean by the latter that the center of mass of the system is at rest or moves with uniform velocity if no external forces are acting on the system. This holds in relativistic mechanics as well, with the proviso that in different Lorentz-frames we must, in general, identify different points as the center of mass. We, however, will always stay in the same frame of reference and so this will not concern us.)

We imagine now two enclosures not subjected to external forces. In each a wave parcel is traveling. In one enclosure part of the path passes through a perfect, nondispersive dielectric (we will simply say glass) where the velocity of propagation is smaller than in empty space. In the other enclosure we have an identical glass rod and an identical wave parcel; its path, however, does not lead through the glass rod. For this reason in the latter enclosure, after time t, the parcel and so the mass associated with its energy would be at a different point than in the first enclosure. Then if we would suppose that the glass rod did not move while the parcel

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[1] M. von Laue, Z. Physik 128, 387 (1950); A. Sommerfeld, "Electrodynamik," *Vorlesungen über theoretische Physik* (W. Klemm, Wiesbaden, 1948), Vol. 3, p. 291.